Straight Lines

The Equation of a Straight Line is y = mx + c

You need to know this stuff like the back of your hand for A-Level — it comes up all over the place, from **logarithms** to **differentiation**.

If you know the **gradient** and the **y-intercept** of a straight line, you can give its **equation**.

> The equation of any straight line can be written in the form $\boldsymbol{y = mx + c}$ where $\boldsymbol{m}$ is the **gradient** of the line and $\boldsymbol{c}$ is the **y-intercept** of the line.

c is the **y-intercept** because it's the point on the graph where $\boldsymbol{x = 0}$ — so $y = m \times 0 + c = c$. Similarly, to find the x-intercept you set y equal to 0 and solve for x.

EXAMPLE: A straight line has a gradient of 4 and a y-intercept of 6. Find the point where this line crosses the x-axis.

First find the equation of the line. The gradient $m = 4$ and y-intercept $c = 6$, so $y = 4x + 6$.
Now set y equal to zero and solve for x: $0 = 4x + 6 \Rightarrow 4x = -6 \Rightarrow x = -1.5$.
So the line crosses the x-axis at the point **(–1.5, 0)**.

You can also read off the gradient and y-intercept from the equation of a line. At **A-Level**, you'll have to deal with straight-line equations written in **different forms**, e.g. $ax + by + c = 0$, rather than $y = mx + c$. Just do a bit of rearranging to get the equation into the form you want.

EXAMPLE: Find the gradient of the line $3x + 4y + 4 = 0$.

Rearrange into $y = mx + c$ form: $4y = -3x - 4 \Rightarrow y = -\frac{3}{4}x - 1$. So the gradient is $\boldsymbol{-\frac{3}{4}}$.

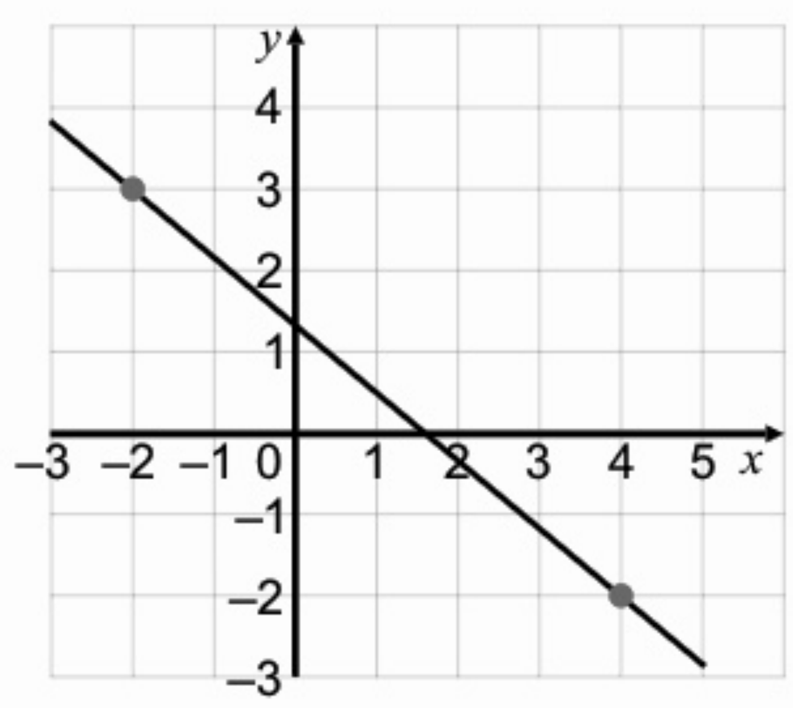
You can Find the Gradient using Two Points on a Line

You can take **any two points on a line** and use their coordinates to find the **gradient** of that line.

The gradient between two points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE: Find the gradient of the straight line on the right.



Pick two accurate points on the line and label them (x_1, y_1) and (x_2, y_2). Here, you can read off the points $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, -2)$.

Plug these into the formula to find the gradient: $\frac{-2-3}{4-(-2)} = \boldsymbol{-\frac{5}{6}}$

x goes to cross, but y's in with the intercept — a strong defensive line...

1) Find the gradient, x-intercept and y-intercept of the line $2x + y = -2$.
2) A line has gradient 3 and y-intercept 5. Find the equation of the line.
3) Find the gradient of the line that goes through the points (3, 4) and (–2, 1).

Straight Lines

Find the *Equation of a Line* using *Two Points* on the line

If you know **two points** that a line goes through, you can work out the **equation** of the line.

EXAMPLE: Find the equation of the line that passes through points (1, 7) and (3, 16). Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

Label the points: $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (3, 16)$

Find the gradient with $\frac{y_2 - y_1}{x_2 - x_1}$: $\frac{16-7}{3-1} = \frac{9}{2}$, so $y = \frac{9}{2}x + c$.

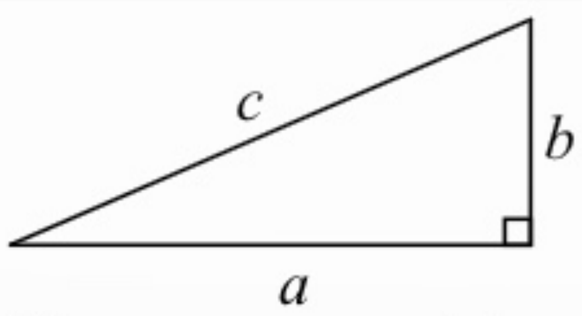
Substitute one of the points into the equation you've found and solve for c:

$y = \frac{9}{2}x + c \Rightarrow 7 = \frac{9}{2} + c \Rightarrow c = 7 - \frac{9}{2} = \frac{5}{2}$. This gives $y = \frac{9}{2}x + \frac{5}{2}$.

Multiply the whole equation by 2 to get integer coefficients: $2y = 9x + 5$.

Then rearrange into the correct form: $\mathbf{9x - 2y + 5 = 0}$

Find the *Distance* between *Two Points* with *Pythagoras*

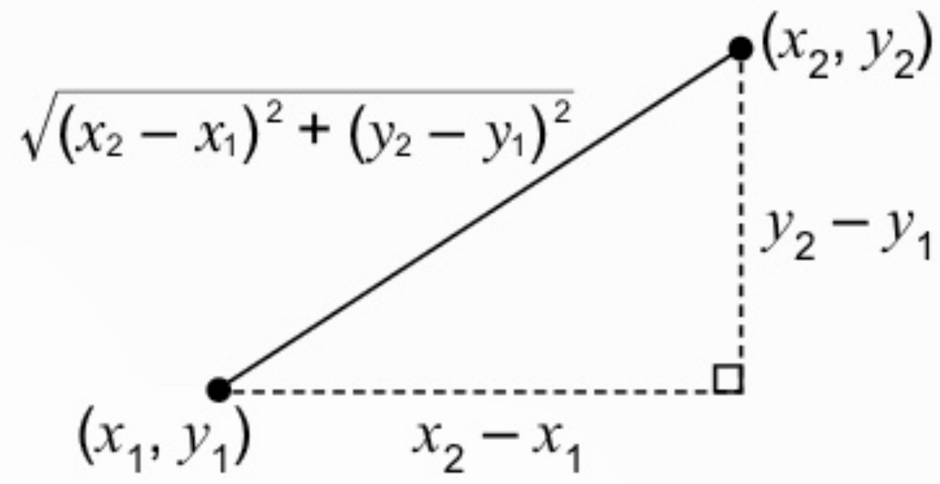


Pythagoras' theorem tells you that, for any **right-angled triangle**, $a^2 + b^2 = c^2$. So **length** $c = \sqrt{a^2 + b^2}$.

At A-Level you'll use this to calculate the magnitude of vectors...

You can use this to find the **distance** between **two points** (x_1, y_1) and (x_2, y_2):

1) Use the two points to make a **right-angled triangle**, with the **hypotenuse** as the line between the two points.
2) The **difference** between the ***x*-values** will give the length of one side. The **difference** between the ***y*-values** will give the length of the other side.



Distance between (x_1, y_1) and $(x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

EXAMPLE: Point A is (4, 17) and point B is (10, 2). What length is line AB?

Label the points: $(x_1, y_1) = (4, 17)$ and $(x_2, y_2) = (10, 2)$, so the length of AB is given by $\sqrt{(10-4)^2 + (2-17)^2} = \sqrt{6^2 + (-15)^2} = \sqrt{261} = \mathbf{16.16}$ (2 d.p.)

You can do a surprising amount with two points...

1) Find the equation of the line that passes through points (1, 2) and (5, 9).
2) A line is drawn between (–3, 4) and (7, –2). Find:
   a) The equation of this line in the form $ax + by + c = 0$ where a, b and c are integers.
   b) The length of the line. Give your answer to 2 d.p.
3) The distance Moe walks is modelled by a straight line graph.
   At time $t = 2$ hours, he has walked a distance of $d = 10$ kilometres.
   At time $t = 3.5$ hours, he has walked a distance of $d = 17.5$ kilometres.
   a) Give the equation of the line in the form $d = mt + c$.
   b) How long does it take Moe to travel 12.5 km?

Parallel and Perpendicular Lines

Parallel Lines have Equal Gradient

At A-Level, you'll need these skills in parts of the differentiation topic, as well as in coordinate geometry.

Parallel lines have the **same gradient**, which means they never cross each other.

EXAMPLE: Line A has equation $y = 3x - 2$. Line B passes through point (4, 3) and is parallel to line A. Find the equation of line B.

Parallel lines have the same gradient, so gradient of line B = gradient of line A = 3.

This gives you the first bit of the equation for line B: $y = 3x + c$.

You know the point (4, 3) is on the line, so plug this into the equation and solve for c:

$3 = 3 \times 4 + c \Rightarrow 3 = 12 + c \Rightarrow c = 3 - 12 = -9$.

So the equation of line B is $\boldsymbol{y = 3x - 9}$.

Perpendicular Lines Cross at Right Angles

You might see perpendicular lines called 'normals' at A-Level.

The **gradients** of perpendicular lines multiply together to give **–1**. So:

The gradient of the perpendicular line = **–1 ÷ the gradient of the other line**

EXAMPLE: Find the equation of the line perpendicular to $3y - 2x + 4 = 0$ that passes through (2, 2). Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

First, rearrange the equation into $y = mx + c$ form: $3y = 2x - 4$ so $y = \frac{2}{3}x - \frac{4}{3}$.

The perpendicular line will have gradient $-1 \div \frac{2}{3} = -1 \times \frac{3}{2} = -\frac{3}{2}$.

Remember — when you divide by a fraction, you flip it over and multiply (see p.7).

So the equation so far is: $y = -\frac{3}{2}x + c$.

Plug in the point (2, 2) and solve for c: $2 = -\frac{3}{2} \times 2 + c$

$2 = -3 + c \Rightarrow c = 2 + 3 = 5$

So the equation of the perpendicular line is $y = -\frac{3}{2}x + 5$.

Finally, rearrange into the form asked for in the question.

Multiply by 2 to get rid of the fractions: $2y = -3x + 10$.

Then put everything on one side of the equation: $\boldsymbol{3x + 2y - 10 = 0}$.

Parallel lines meet about as often as the bad guy wins in Bond films...

1) Line F has equation $5y + 3x - 7 = 0$. Line G is parallel to line F and goes through the origin. What is the equation of line G?
2) Lines P and Q are perpendicular and meet at the point (–1, –2). The gradient of line P is 4. Find the equations of P and Q. Give your answers in the form $ax + by + c = 0$.
3) The points P (4, 3) and Q (9, 5) lie on line l_1. The line l_2 is perpendicular to l_1 and passes through Q. Find the equation of line l_2 in the form $ax + by + c = 0$.

Quadratic Graphs

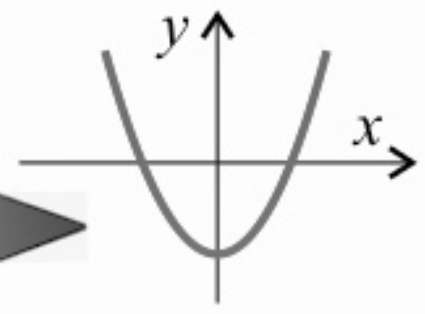
All Quadratic Graphs have a *Symmetrical Bucket Shape*

You'll need to be really confident with **sketching quadratics** at A-Level — there's a whole topic on it. You'll also use these skills in the **inequalities** and **kinematics** topics.

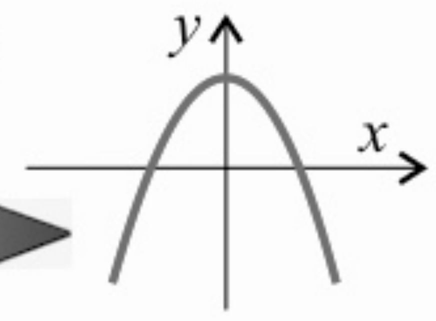
To remember, think 'n' for negative...

All **quadratic** graphs have the **same shape** — a **symmetrical curve**.

Positive quadratics (ones where the coefficient of x^2 is positive) are **u-shaped** like this.



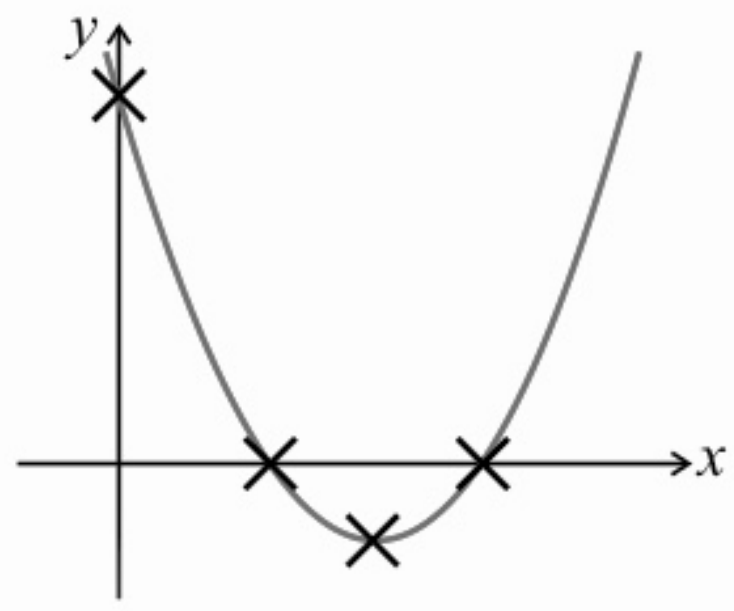
Negative quadratics (ones where the coefficient of x^2 is negative) are **n-shaped** like this.



Work Out *Three Things* to *Sketch A Quadratic*

If you're asked to **sketch** a quadratic, you **don't** need to use graph paper and carefully plot every point. You just need to work out and label the **most important** bits of the graph to make sure your drawing is **roughly correct**. As well as the shape of the graph, you need to work out:

1) Where the graph **crosses** the ***y*-axis** (the ***y*-intercept**).
2) Where the graph **crosses** the ***x*-axis** (the ***x*-intercepts**). These are called the **roots** of the quadratic.
3) You might be asked to find the **turning point** (or '**vertex**') of the graph — this could be the **minimum** or **maximum** point depending on whether it's **u-shaped** or **n-shaped**.



EXAMPLE: Sketch the graph of $y = x^2 - 5x + 6$, including any points of intersection with the axes and the vertex of the graph.

If the quadratic won't factorise easily, you might need to complete the square or use the quadratic formula (see p.18-21).

The coefficient of x^2 is positive, so the graph is **u-shaped**.

To find the y-intercept, let $x = 0$: $y = 0^2 - 5 \times 0 + 6 = 6$. So it **crosses the *y*-axis** at $\boldsymbol{y = 6}$.

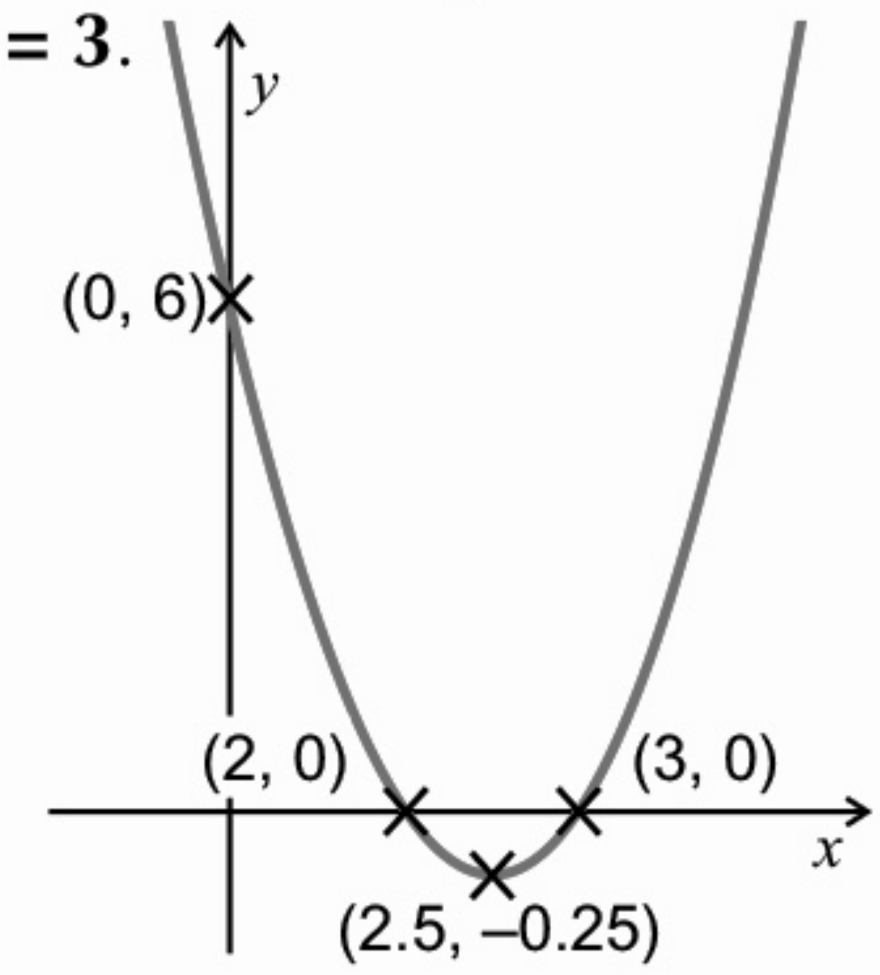
The graph intersects the x-axis when $y = 0$, so solve $x^2 - 5x + 6 = 0$ by factorising:

$x^2 - 5x + 6 = (x - 2)(x - 3) = 0$, so the ***x*-intercepts** are $\boldsymbol{x = 2}$ and $\boldsymbol{x = 3}$.

Quadratic graphs are symmetrical, so the ***x*-coordinate** of the **turning point** of the graph is halfway between 2 and 3. So it's $(2 + 3) \div 2 = \mathbf{2.5}$.

Plug this number back into the quadratic to find the ***y*-coordinate** of the **turning point**: $2.5^2 - 5 \times 2.5 + 6 = \mathbf{-0.25}$. So the **turning point** is **(2.5, –0.25)**.

Now you can sketch the graph. Make sure to label each of the points.



If a graph has a '**double**' **root** (i.e. factorises to the form $(x + a)^2$), this root is the **vertex**. The graph just **touches** the ***x*-axis** at this point — it doesn't cross it.

Symmetrical bucket hats — this year's must-have maths fashion trend...

1) Sketch these quadratics, labelling the points of intersection with the axes and the vertex of each graph.

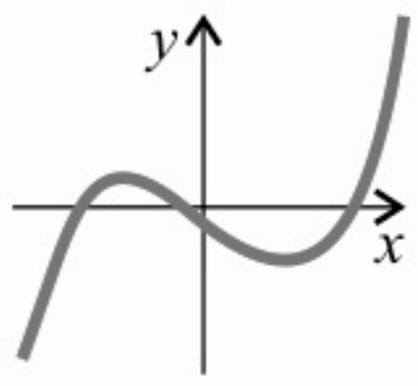
a) $y = x^2 - 3x$ b) $y = x^2 + x - 2$ c) $y = -x^2 + 6x - 9$ d) $y = 3x^2 + 2x - 8$

Harder Graphs

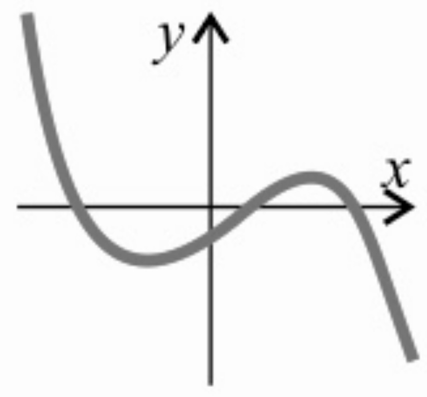
A **Cubic** Contains an x^3 Term

Cubics are of the form $ax^3 + bx^2 + cx + d$ where a, b, c and d are all numbers ($a \neq 0$).
Cubic graphs have a characteristic '**wiggly**' shape.

If x^3 has a **positive** coefficient, then the graph goes from the **bottom left** to the **top right** of the axes.



If x^3 has a **negative** coefficient, then the graph goes from the **top left** to the **bottom right**.



At **A-Level** you'll learn how to **factorise** and **sketch** simple cubics.
Here's an example of how to sketch one.

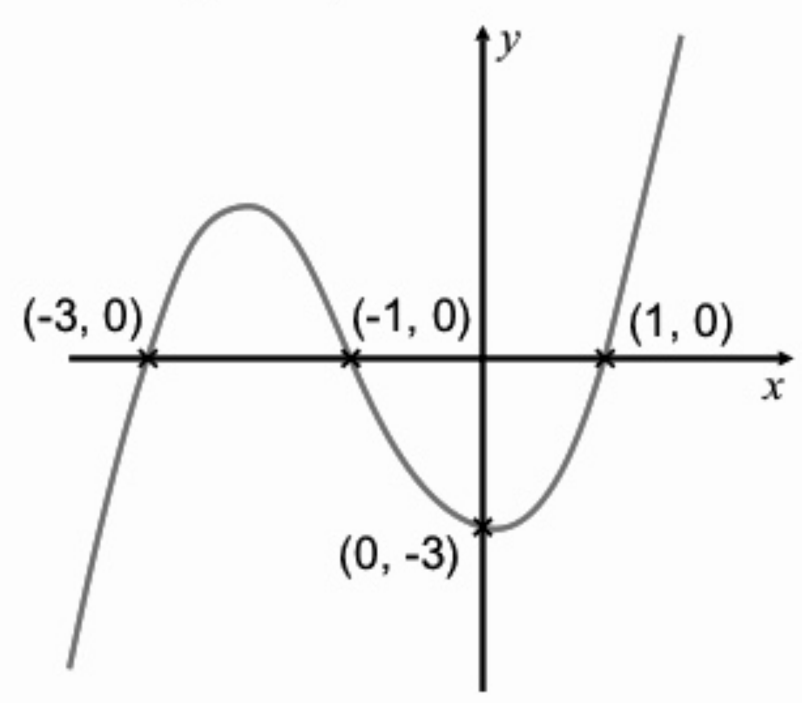
EXAMPLE: Given that $x^3 + 3x^2 - x - 3 = (x + 1)(x - 1)(x + 3)$, sketch the graph of $y = x^3 + 3x^2 - x - 3$.

The method for sketching a cubic is similar to the method for sketching a quadratic.

The x^3 term has a positive coefficient, so the graph will go **from the bottom left to the top right**.

Now you just need to find where the graph crosses the axes. It **crosses the *y*-axis** when $x = 0$, so at $\boldsymbol{y = -3}$. It crosses the x-axis when $y = 0$: $x^3 + 3x^2 - x - 3 = (x + 1)(x - 1)(x + 3) = 0$, so it **crosses the *x*-axis** at $\boldsymbol{x = -1}$, **1** and **–3**.

Now you can use these values to sketch the graph.



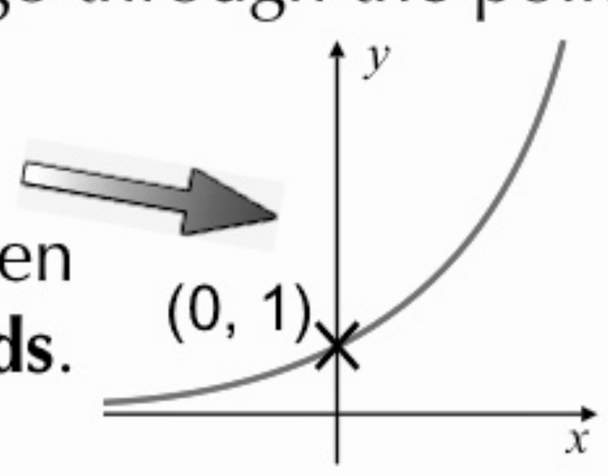
Exponential Graphs have Equation $y = k^x$ or $y = k^{-x}$

You'll cover lots on exponentials at A-Level.

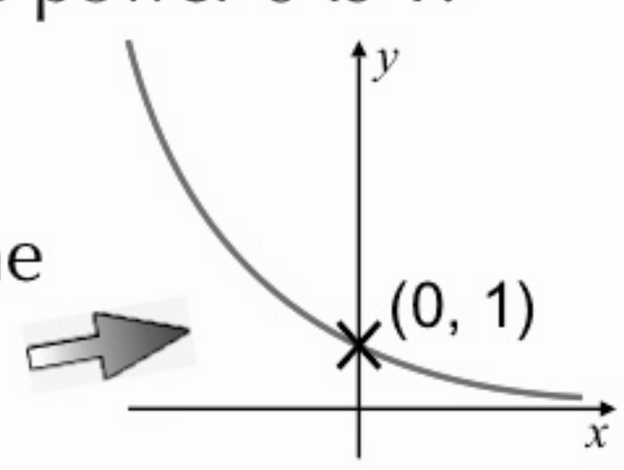
Graphs of the form $y = k^x$ or $y = k^{-x}$ (for any positive number k) are called **exponential** graphs. An exponential graph is a **curve** that is always **above** the ***x*-axis**.

All exponential graphs go through the point **(0, 1)** — because anything to the power 0 is 1.

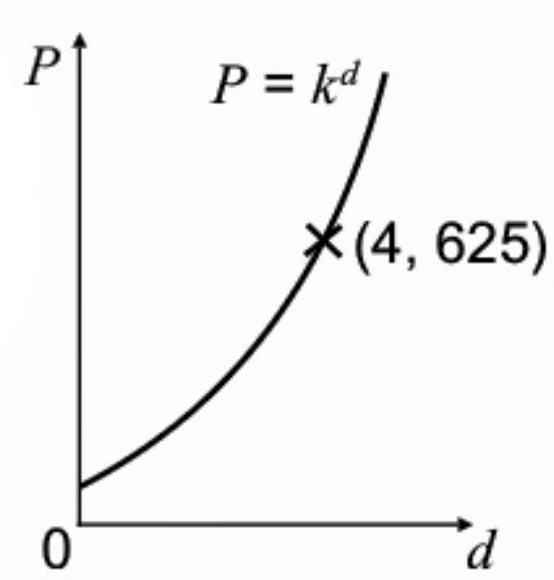
If k is **bigger than 1** and the power is **positive**, then the graph **curves upwards**.



If k is **between 0 and 1**, **OR** the **power is negative**, then the graph is **flipped horizontally**.



EXAMPLE: This graph shows how the size of a population of bacteria (P) changes over time. The graph has equation $P = k^d$, where d is time in days and k is a positive constant. Find k.



From the graph you can see that when $d = 4$, $P = 625$.
Substitute these values into the equation to find k:

$P = k^d \Rightarrow 625 = k^4 \Rightarrow k = \sqrt[4]{625} = \mathbf{5}$

What goes up must come down — unless you're an exponential graph...

1) Sketch the graphs of the following cubics:

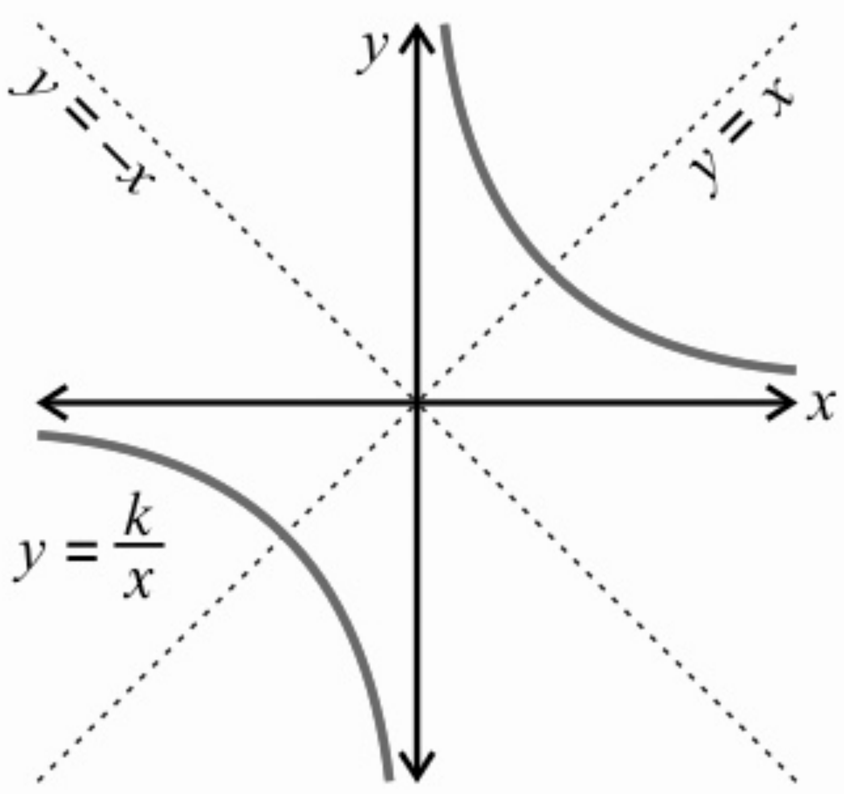
a) $y = (x + 1)(x - 1)(x + 2)$ b) $y = (x - 1)(x - 2)(x - 3)$ c) $y = -x^3 - 5x^2 + 6x$

Harder Graphs

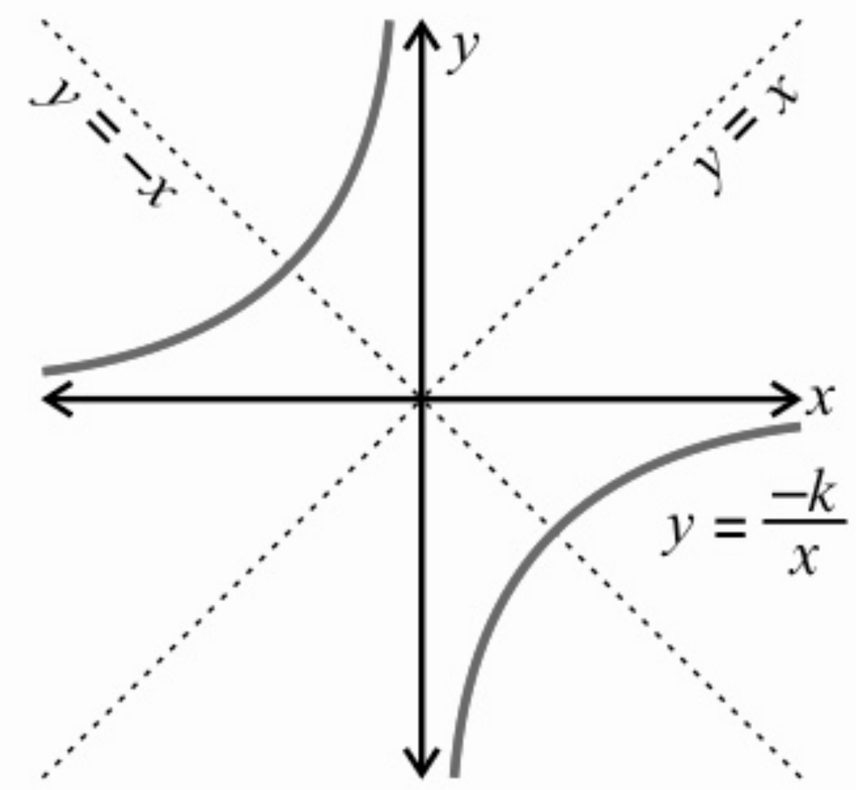
Reciprocal Graphs have Equation *y = k/x* or *xy = k*

1) Reciprocal graphs have **two curves** in **diagonally opposite quadrants**. They're **symmetrical** about the lines $y = x$ and $y = -x$.
2) The pair of quadrants the curves are in depends on whether k is **positive** or **negative**.

When k is **positive**, the graph looks like **this**.



When k is **negative**, the graph looks like this.



3) **Reciprocal graphs** are undefined at $x = 0$ (the **y-axis**) and $y = 0$ (the **x-axis**).
4) So reciprocal graphs **never touch** the axes — but they do get **infinitely close**. This means the axes are **asymptotes** of the graph — you'll be expected to know this term at A-Level.

Circles have Equations $(x - a)^2 + (y - b)^2 = r^2$

The general equation for a **circle** with **centre** (a, b) and **radius** r is $(x - a)^2 + (y - b)^2 = r^2$.
Circle equations of the form $x^2 + y^2 = r^2$ were covered at GCSE.
These circles have centre (0, 0), so are the **special case** where $a = b = 0$.

EXAMPLE: Find the equation of the tangent to $(x - 5)^2 + (y - 5)^2 = 50$ at the point (–2, 4).

From the general equation of a circle, you know the centre is (5, 5). Start by finding the gradient of the radius — the line from the centre of the circle to the point (–2, 4).

Gradient of radius $= \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{5-4}{5-(-2)} = \dfrac{1}{7}$.

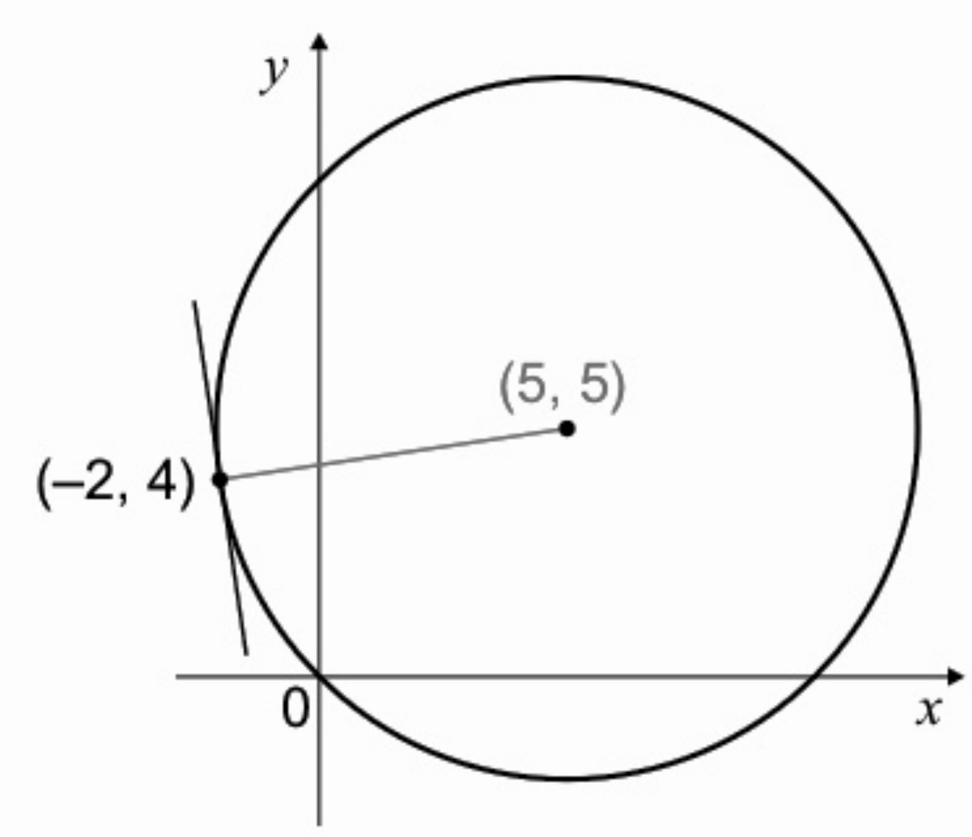
A tangent to a circle meets a radius at 90°, so they are perpendicular (this is one of the circle theorems from GCSE).

This means the gradient of the tangent is $-1 \div \dfrac{1}{7} = -7$.

So the equation of the tangent so far is $y = -7x + c$.

Substitute in the point (–2, 4) and solve for c:
$4 = -7 \times -2 + c \Rightarrow 4 = 14 + c \Rightarrow c = -10$.

So the equation of the tangent is $\boldsymbol{y = -7x - 10}$.



See page 36 for more on perpendicular lines.

Reciprocal graphs — a good way to get one over on your x...

1) Give the centre and radius of the following circles:
   a) $(x - 1)^2 + (y - 3)^2 = 16$ b) $x^2 + (y + 2)^2 = 50$ c) $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = 2$
2) Find the equation of the tangent to the circle $(x - 1)^2 + (y - 2)^2 = 8$ passing through point (3, 4).

Graph Transformations

f(x) + a is a Translation Along the y-axis

A transformation of the form $f(x) + a$ is a **translation parallel** to the ***y*-axis**. The graph of $f(x)$ moves ***a* units** in the y-direction (i.e. vertically), so all of the ***y*-values** of $f(x)$ will have ***a* added to them**. Translating a graph doesn't change its shape, it just **moves** it.

Make sure you're really confident with all these types of transformation — you'll learn a few more at A-Level.

EXAMPLE: The graph of $y = f(x)$ has a minimum point at $(2, -4)$. Write down the coordinates of the minimum point of $f(x) + 5$.

This is a translation along the y-axis — so it only affects the y-values.
This means you add 5 onto the y-value of the minimum point.
So the minimum of $f(x) + 5$ is $(2, -4 + 5) =$ **(2, 1)**.

At A-Level, you might have to describe a translation using a **column vector**.
This type of translation can be described by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

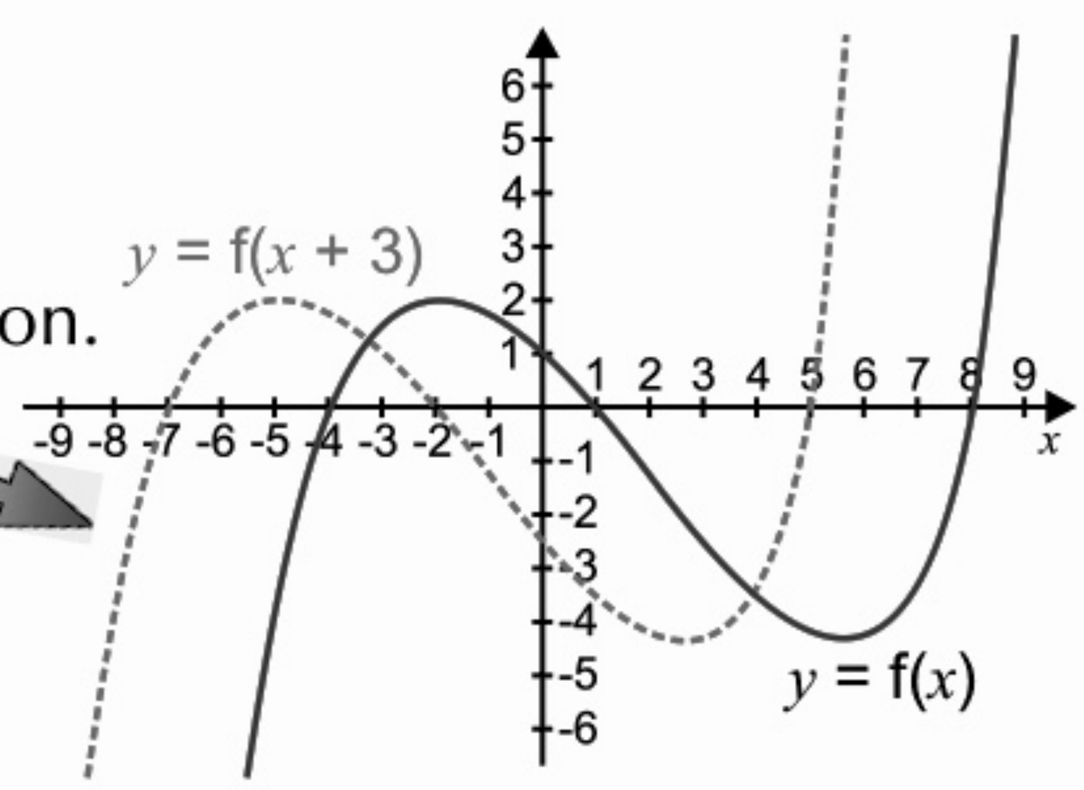
f(x + a) is a Translation Along the x-axis

A transformation of the form $f(x + a)$ moves the graph of $f(x)$ **left** or **right** — i.e. **parallel** to the ***x*-axis**.
Be **careful** with these — when a is **positive**, the graph moves to the **left**. When it is **negative**, the graph moves to the **right**.

EXAMPLE: The diagram shows the graph of $y = f(x)$. Sketch the graph of $y = f(x + 3)$.

$a = 3$, so the graph moves 3 units in the negative x-direction.
This means the graph moves to the left by three units.

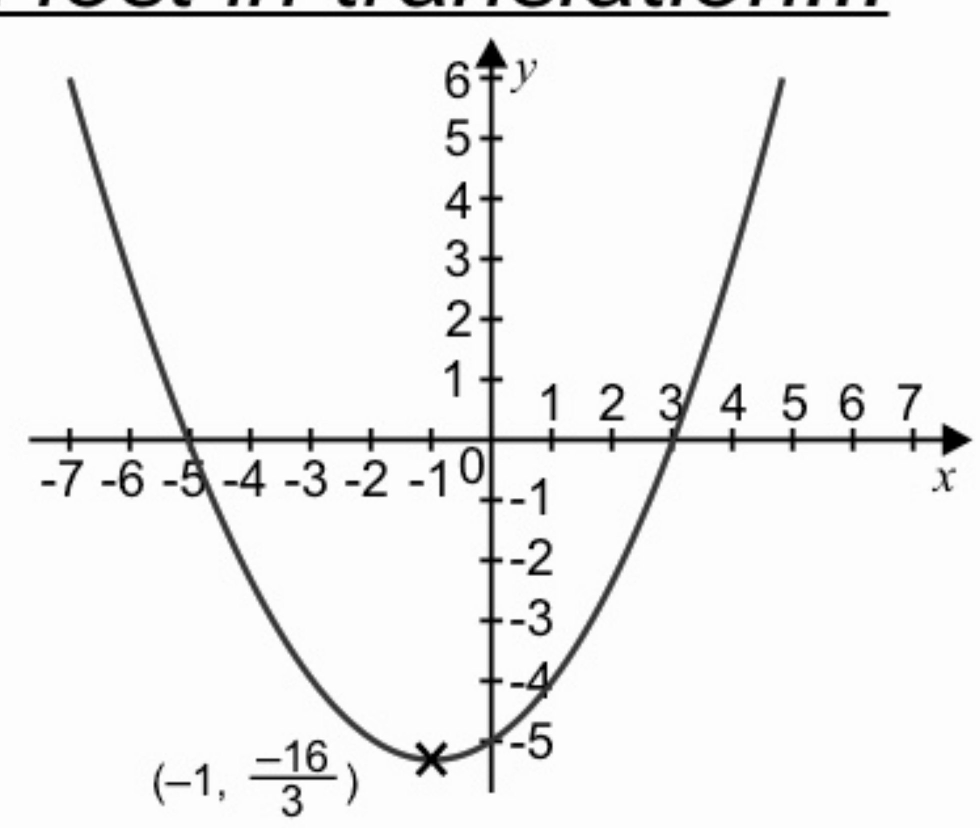
Even if you're given a graph that you don't recognise, just remember how the translations work and you'll be fine.



This type of translation can be described by the column vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

I can't find that graph anywhere — it must've been lost in translation...

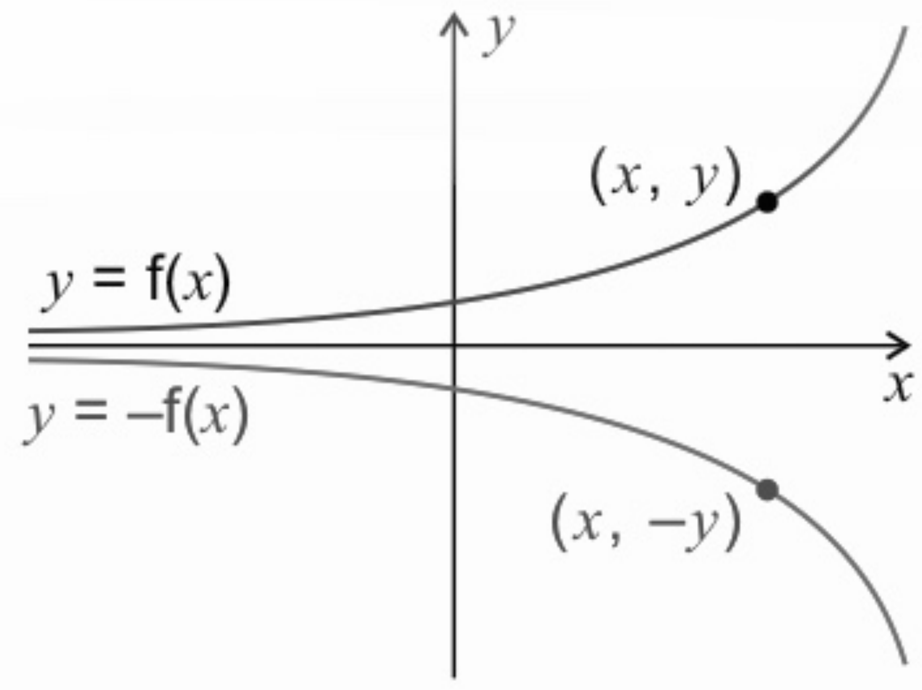
1) The diagram on the right shows the graph of $y = f(x)$:
   a) Sketch the following translations:
      i) $f(x) + 3$, labelling the coordinates of the minimum and where the curve meets the y-axis.
      ii) $f(x - 2)$, labelling the coordinates of the minimum and where the curve meets the x-axis.
   b) State the column vectors that describe the above translations.



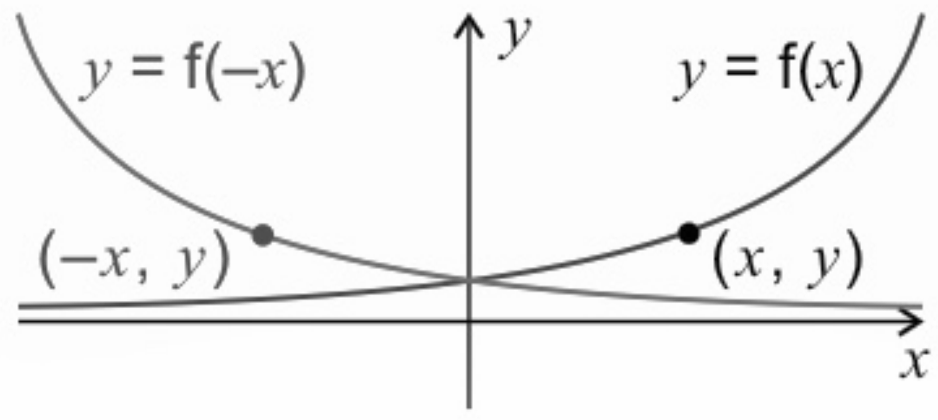
Graph Transformations

–f(x) and f(–x) are both Reflections

$y = -f(x)$ is the **reflection** of $y = f(x)$ in the **x-axis**.
For **every point** on the graph of $y = f(x)$, the **x-coordinate** stays the **same** and the **y-coordinate** is **multiplied** by **–1**.



$y = f(-x)$ is the **reflection** of $y = f(x)$ in the **y-axis**.
For **every point** on the graph of $y = f(x)$, the **y-coordinate** stays the **same** and the **x-coordinate** is **multiplied** by **–1**.



EXAMPLE: a) $f(x) = x^2 + 2x - 8$.
Find the y-intercept of $y = -f(x)$.

The y-intercept of $f(x)$ is when $x = 0$.
$f(0) = (0)^2 + 2(0) - 8 = -8$, so is at $(0, -8)$.

The y-intercept of $y = -f(x)$ is the y-intercept of $y = f(x)$ multiplied by -1.
So $y = -f(x)$ intersects the y-axis at **(0, 8)**.

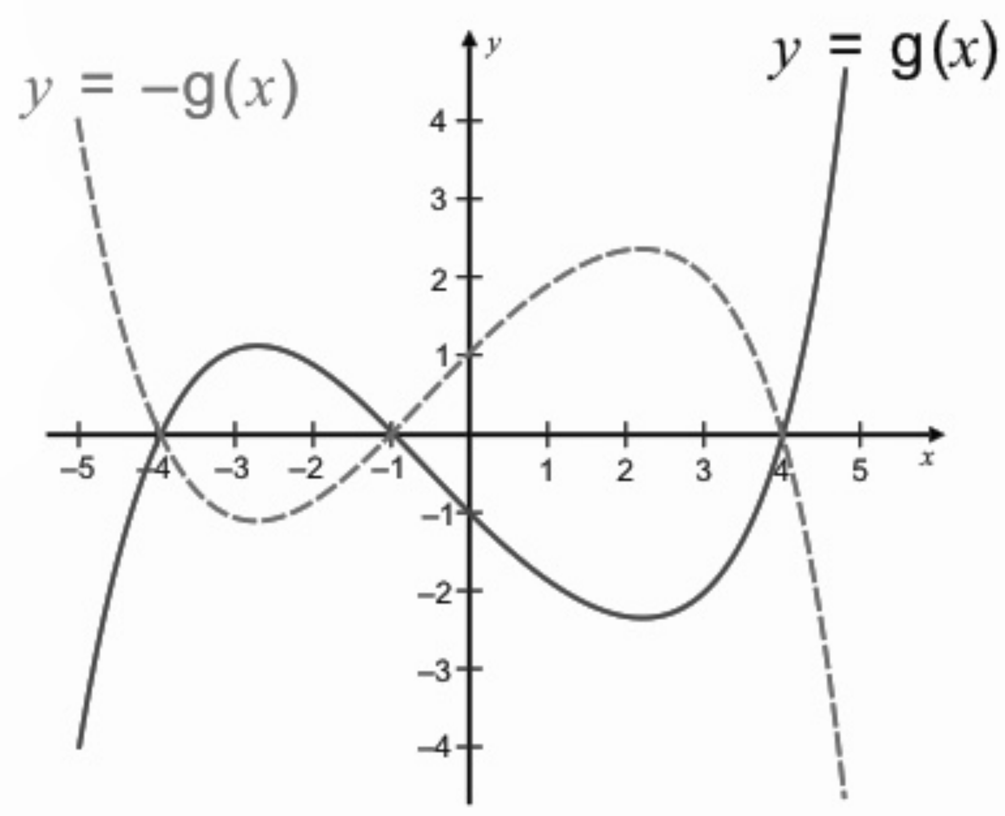
b) Given that $f(x) = (x - 2)(x + 4)$, find the roots of $f(-x)$.

The roots of $f(x)$ are $x = 2$ and $x = -4$.
$f(-x)$ is a reflection in the y-axis, so all the x-values are multiplied by -1.
This means the roots of $f(-x)$ are **$x = -2$** and **$x = 4$**.

EXAMPLE: The diagram shows the graph of $y = g(x)$.
Sketch the graph of $y = -g(x)$.

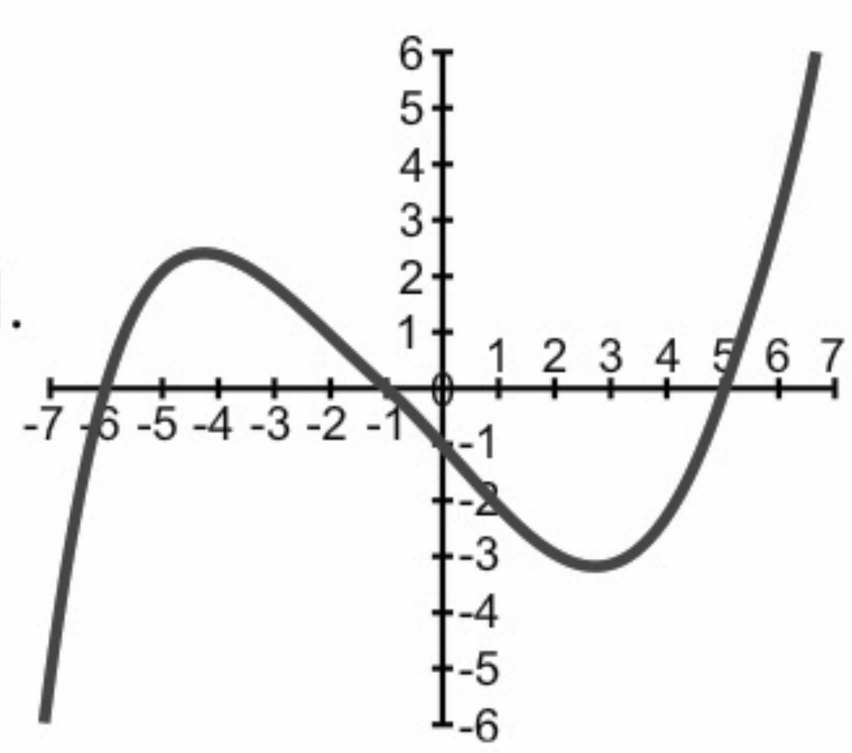
The graph of $y = -g(x)$ is a reflection of $y = g(x)$ in the x-axis.

The x-values stay the same, but the y-values are multiplied by -1. This means the y-intercept is 1 instead of -1.



Now would be a good time to stop and reflect on how far we've come...

1) The diagram on the right shows the graph of $y = f(x)$.
Sketch the graphs of $y = -f(x)$ and $y = f(-x)$, labelling the x- and y-intercepts of each.



2) The quadratic graph $y = g(x)$ and its reflection $y = g(-x)$ are identical.
For each of the following, state whether it could be $g(x)$.

a) $x^2 - 4$ b) $x^2 + 3x - 1$ c) $-x^2$

3) $y = h(x)$ is the graph of a quadratic with the turning point $(4, -7)$.
State the coordinates of the turning point of the following graphs:

a) $y = -h(x)$ b) $y = h(-x)$

Trigonometry — Sin, Cos, Tan

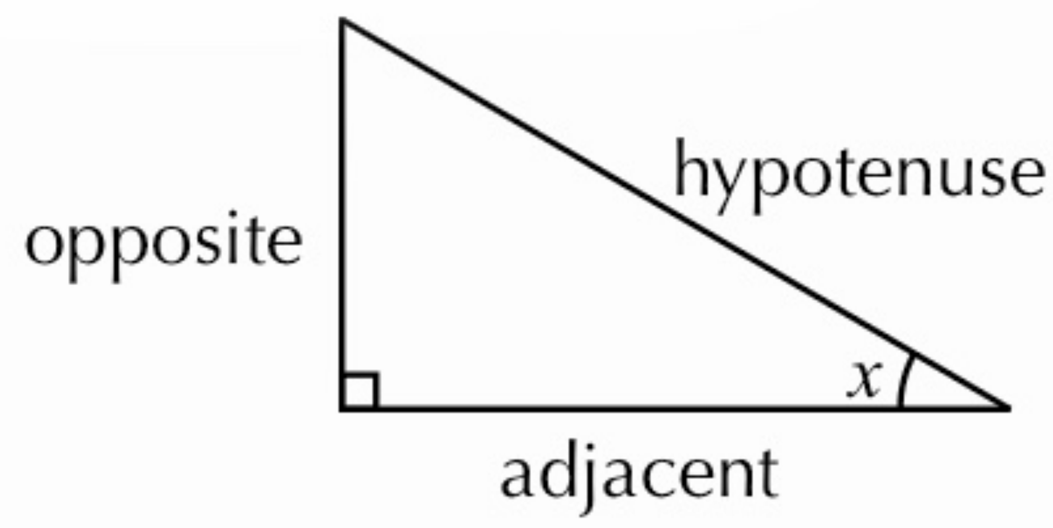
Use SOHCAHTOA to Remember how to use Sin, Cos and Tan

Remember SOH CAH TOA for working out side lengths and angles in right-angled triangles:

At A-Level, you'll come across a lot more trigonometry, so you need to have a good grip on the basics.

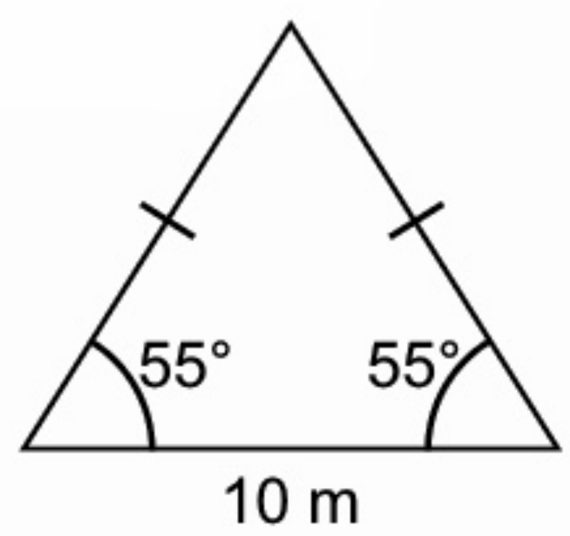
$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos x = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan x = \frac{\text{opposite}}{\text{adjacent}}$$

- The **hypotenuse** is the longest side of the triangle.
- The **opposite** is the side opposite the angle you're working with (x).
- The **adjacent** is the side next to the angle you're working with (x).

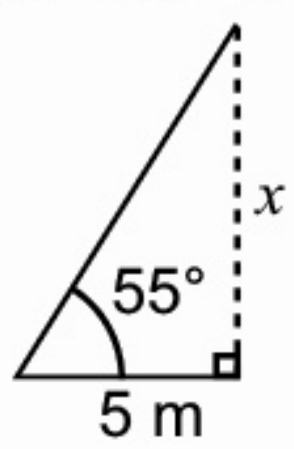


Which formula you need depends on the sides and angles you're given in the question.

EXAMPLE: Calculate the height of this isosceles triangle to 3 s.f.



Start by splitting the triangle down the middle to get two right-angled triangles. Then you can use trigonometry on one of them.

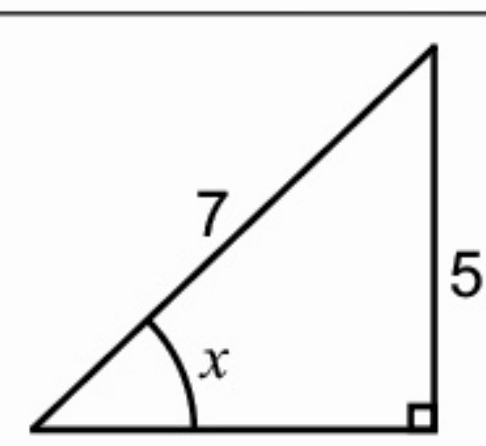


You have the adjacent side and want to find the opposite side, so use tan:

$\tan 55° = \frac{x}{5} \Rightarrow x = 5 \times \tan 55° = 7.1407... =$ **7.14 m (3 s.f.)**

To calculate a missing angle, you need to use one of the **inverse trig functions** — $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$.

EXAMPLE: Find angle x to 3 s.f.

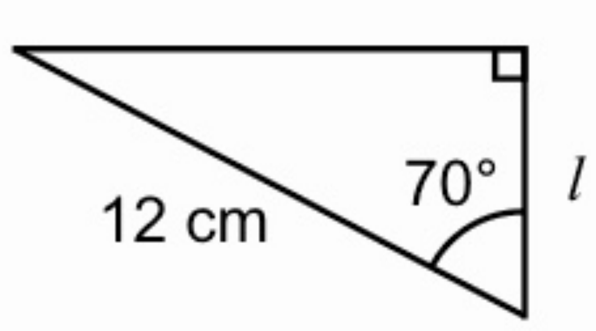


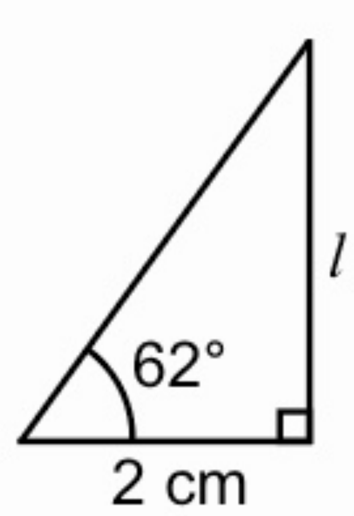
You know the opposite and the hypotenuse, so you need to use sin:

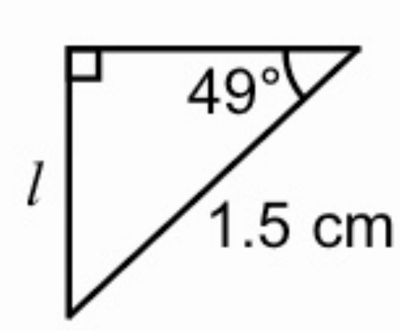
$\sin x = \frac{5}{7} \Rightarrow x = \sin^{-1}\left(\frac{5}{7}\right) = 45.5846... =$ **45.6° (3 s.f.)**

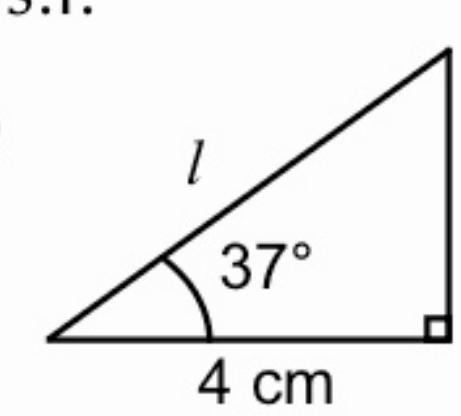
Maths rap group idea #23 — The Wu Tan(x) Clan...

1) Find the missing length l for each of these triangles. Give your answer to 3 s.f.

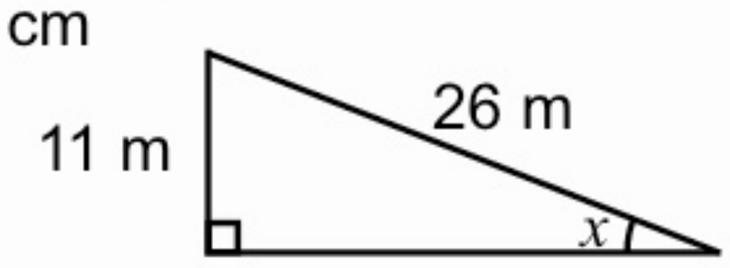
a) 

b) 

c) 

d) 

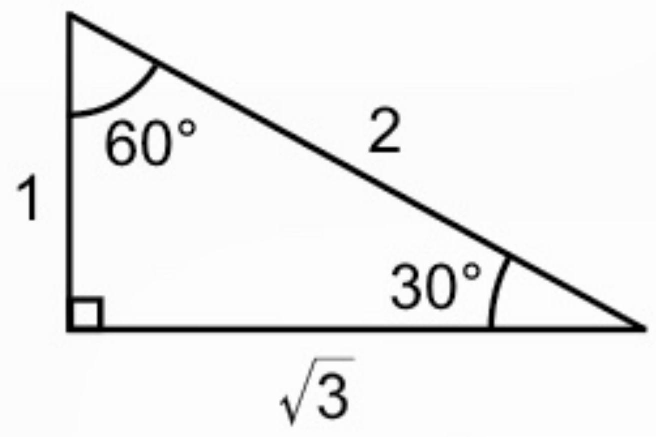
2) Calculate the missing angle x to 2 d.p.

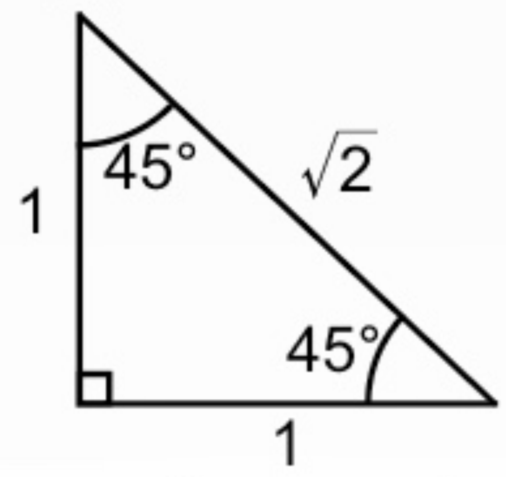


Trigonometry — Sin, Cos, Tan

Learn these Trig Values

There are **two triangles** that you can use to help you work out important trig values:





You can use Pythagoras' Theorem to check that the side lengths are right (p.35).

Using SOH CAH TOA on these triangles gives you these values:

$\sin 30° = \frac{1}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$ $\quad$ $\sin 45° = \frac{1}{\sqrt{2}}$

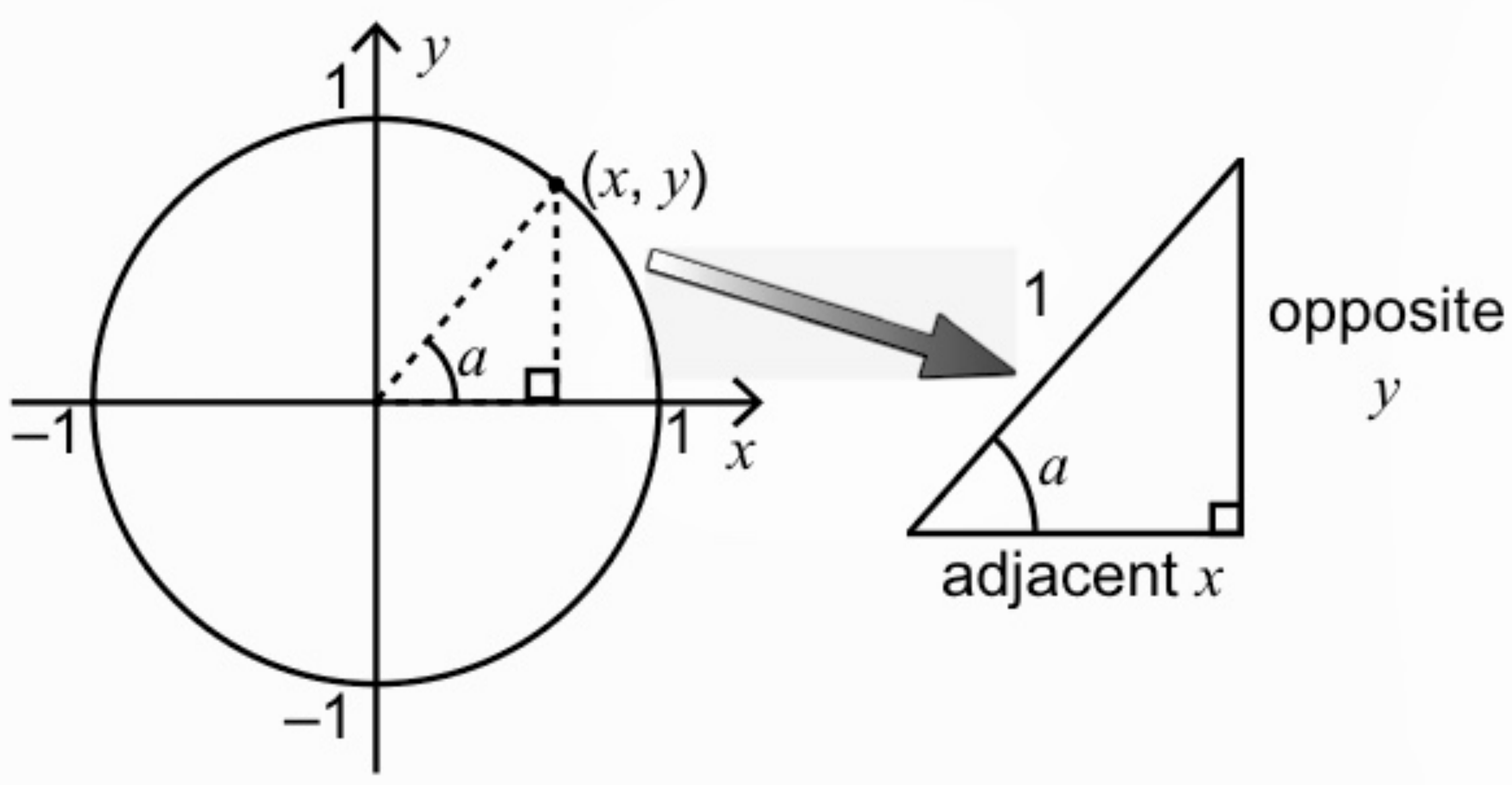
$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 60° = \sqrt{3}$ $\quad$ $\tan 45° = 1$

*You can **Calculate Trig Values** using the **Unit Circle***

The **unit circle** is a circle with **radius 1** centred on the **origin**.
At A-Level, you'll see the unit circle used to calculate trig values.

1) You can take a **point** on the unit circle and make a **right-angled triangle**.
2) You know the **hypotenuse** of this triangle will be **1**, as it is the **radius** of the unit circle.

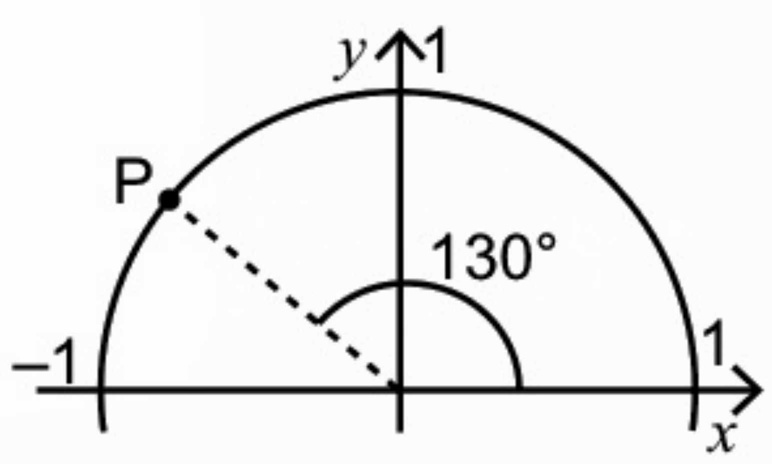


This shows that:
$\cos a = \frac{x}{1} = x$
$\sin a = \frac{y}{1} = y$

When angle a is 0°, the point on the circle is (1, 0), so cos 0° = 1 and sin 0° = 0. When angle a is 90°, the point on the circle is (0, 1), so cos 90° = 0 and sin 90° = 1.

3) This means you can find the **coordinates** of any point **(x, y)** on the unit circle by using trig — the coordinates are given by **(cos a, sin a)**.
4) The angle a needs to be measured **anticlockwise** from the **positive x-axis**.

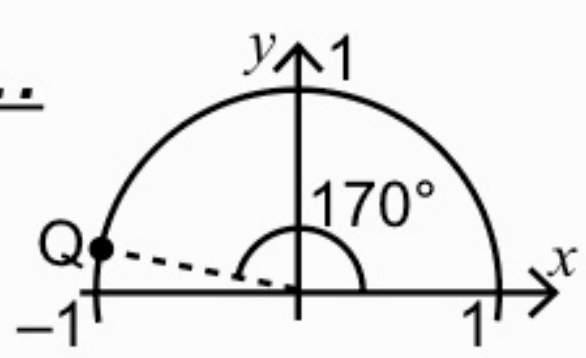
EXAMPLE: Find the coordinates of point P on the unit circle. Give your answer to 2 d.p.



The point is on the unit circle, so you know the x-coordinate is cos 130° = –0.64278... and the y-coordinate is sin 130° = 0.76604...)
So the coordinates of P to 2 d.p. are **(–0.64, 0.77)**

More circles and triangles than a PlayStation® cheat code...

1) The diagram on the right shows a point Q that lies on the unit circle. What are the coordinates of Q? Give your answer to 2 d.p.

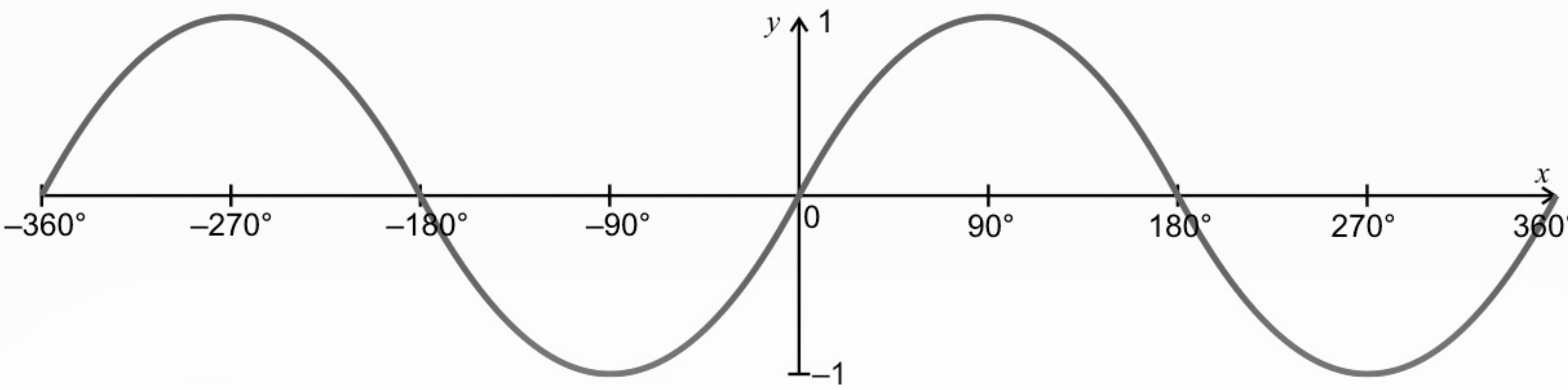


Trigonometry — Graphs

The Graph of y = sin x

You should have seen **all three** of the trig graphs at GCSE. You'll need to be really comfortable with them at A-Level, because they're used for **solving trig equations**.

This is the graph of $y = \sin x$:



1) The graph bounces **between $y = -1$ and $y = 1$**. All the values of $\sin x$ are **within** this **range**.
2) $\sin 0° = 0$, so the graph goes through the origin. It also crosses the x-axis every 180°.
3) The basic shape of the graph **repeats every 360°**. This means that the values of $\sin x$ repeat every 360°, so $\sin x = \sin (x + 360°) = \sin (x - 360°)$.
4) The graph has **rotational symmetry** about the **origin** — if you rotated it 180° around (0, 0), it would look the **same**. This means that **$\sin (-x) = -\sin x$**.

If a graph repeats, it's a 'periodic graph'. Because $\sin x$ repeats every 360°, you say it has a period of 360°.

On the next few pages, you'll see how to use trig graphs to solve equations — like you will at A-Level. Here's an example with $y = \sin x$.

If the first solution you find is outside the interval you're interested in, add or subtract multiples of 360° to find a solution inside the interval.

EXAMPLE: Solve $\sin x = 0.5$ for $0° \leq x \leq 360°$.

This is one of the trig values you should know: $x = \sin^{-1} (0.5) = 30°$

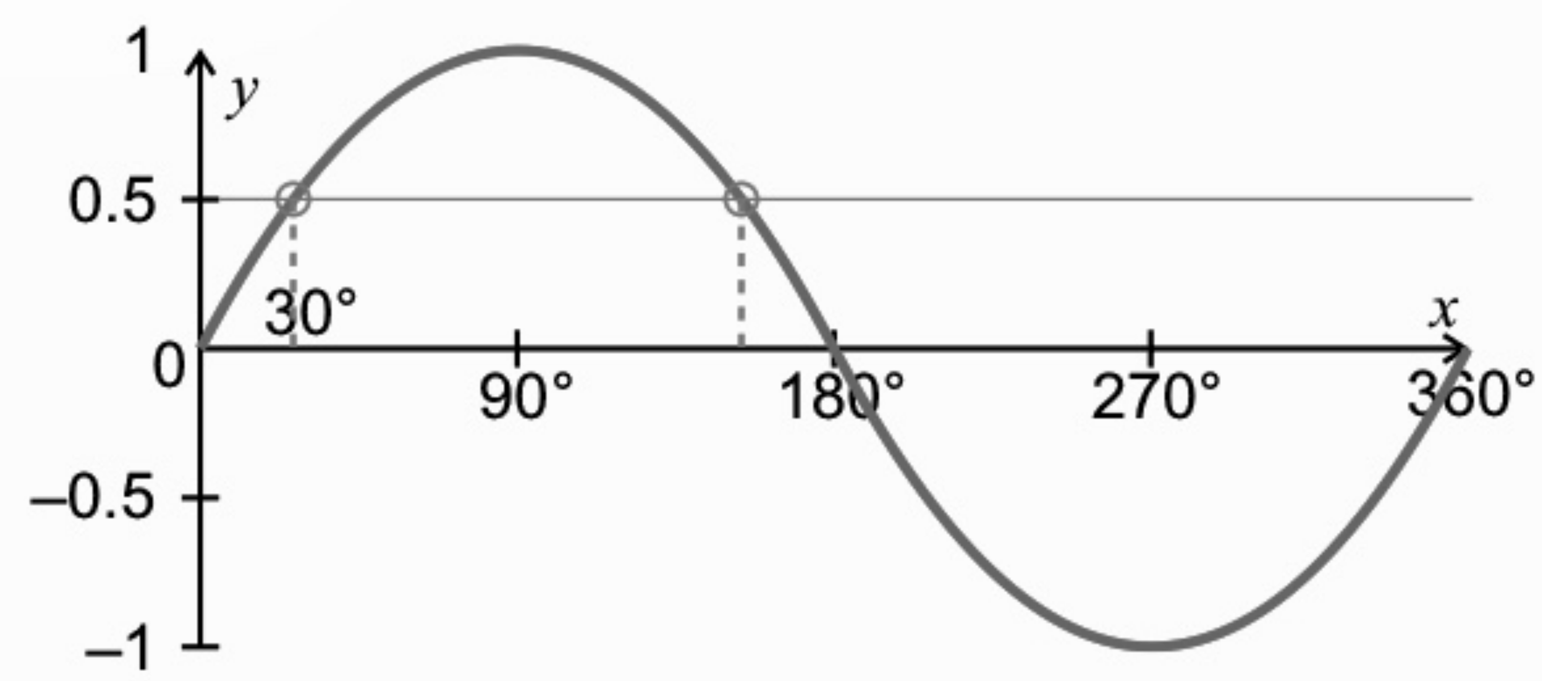
Sketch the graph of $y = \sin x$ to find the other solution:

Draw on a line for $y = 0.5$

Each place that this line meets the sin curve will give a solution.

The first solution is the one you've already found — $x = 30°$.

To find the other one you use the symmetry of the graph.



The first solution is 30° away from the origin, so you know that the other solution is 30° away from 180°: $180° - 30° = 150°$

So the solutions to $\sin x = 0.5$ for $0° \leq x \leq 360°$ are $\boldsymbol{x = 30°}$ and $\boldsymbol{x = 150°}$

It'd be a sin not to learn every last bit of this page...

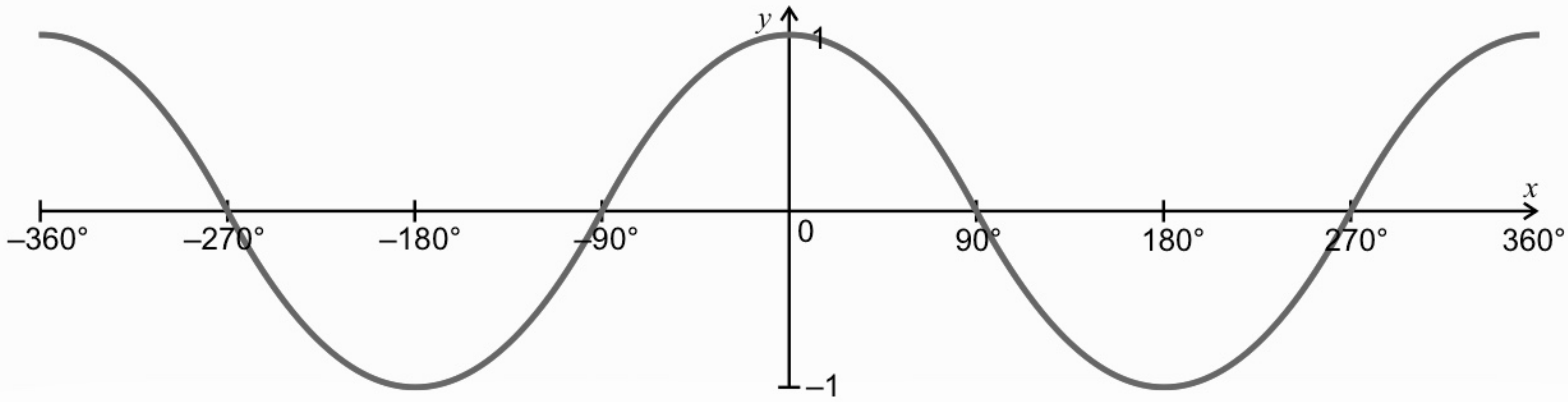
1) By sketching a graph, find all the solutions to the following equations in the interval $0° \leq x \leq 360°$. Where necessary, give your answers to 1 decimal place.

a) $\sin x = \dfrac{1}{\sqrt{2}}$ b) $\sin x = -0.7$ c) $\sin x = -0.2$

Trigonometry — Graphs

The Graph of y = cos x

The graph of **cos *x*** is the **same shape** as the graph of **sin *x***, but shifted left by 90°.
So the graph of $y = \cos x$ looks like this:



Because it's the same shape as the sin graph, it shares some of the same properties:

1) The ***y*-values** for the graph are all **between –1 and 1**.
2) The basic shape of the graph **repeats every 360°**, so $\cos x = \cos(x + 360°) = \cos(x - 360°)$.

But there are some differences between the two:

1) cos *x* is **symmetrical** about the *y*-axis — you can **reflect** it in the *y*-axis and it'll look the same. So $\cos(-x) = \cos(x)$.
2) Because $\cos 0° = 1$, the graph crosses the *y*-axis at $y = 1$.
3) It **crosses the *x*-axis** at **± 90°**, **± 270°** etc.

Just like with sin *x*, if the first solution you find is outside the interval, add or subtract multiples of 360° to find a solution inside the interval.

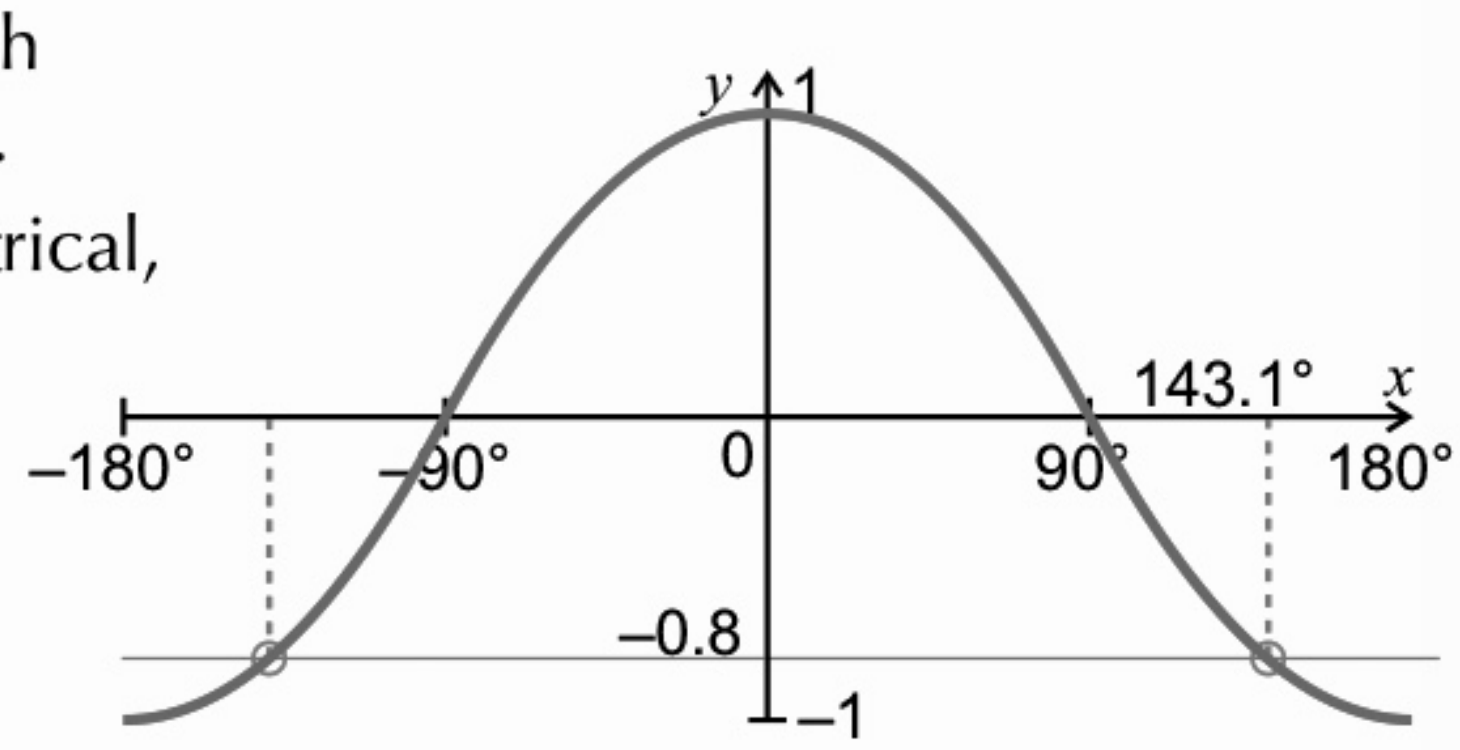
EXAMPLE: Solve $\cos x = -0.8$ for $-180° \le x \le 180°$. Give your answers to 1 d.p.

Solving $\cos x = -0.8$ with a calculator will give you $x = \cos^{-1}(-0.8) = 143.1301...°$

This gives one answer. Now sketch the graph over this interval to find the second solution.

As every 360° interval of the graph is symmetrical, the second solution will be the same distance to the left of the *y*-axis as the first is to the right.

The first solution is 143.1301...° to the right of the *y*-axis, so the second solution is 143.1301...° to the left of the *y*-axis. So $x = -143.1301...°$



So the solutions to $\cos x = -0.8$ for $-180° \le x \le 180°$ are $\boldsymbol{x = 143.1°}$ **and** $\boldsymbol{x = -143.1°}$ **(1 d.p.)**

Cosimodo — The humpback trig graph of Notre Dame...

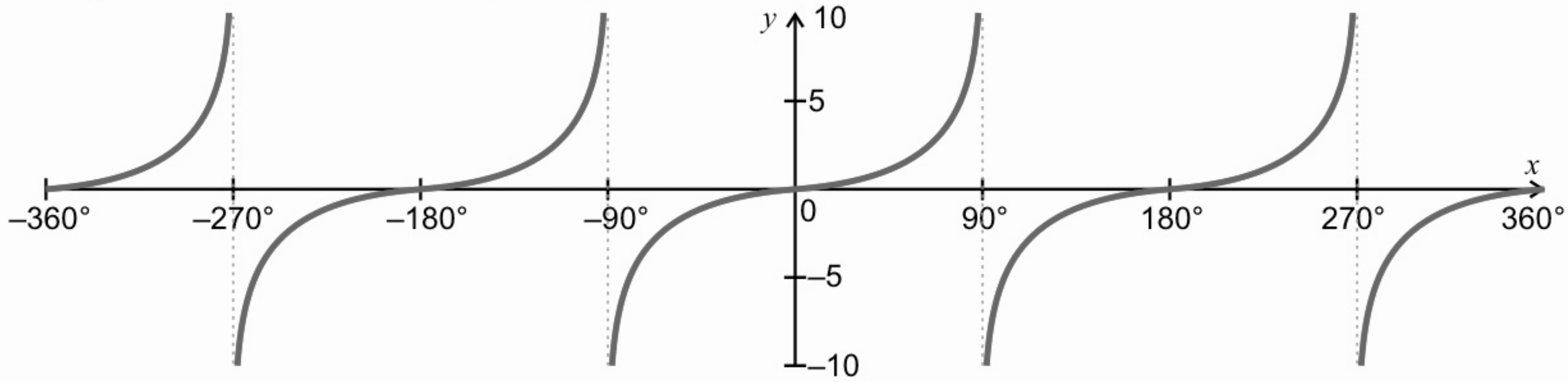
1) By sketching a graph, find all the solutions to the following equations in the interval $-360° \le x \le 360°$. Where necessary, give your answers to 1 decimal place.

a) $\cos x = \frac{1}{\sqrt{2}}$ b) $\cos x = 0.1$ c) $\cos x = -0.4$

Trigonometry — Graphs

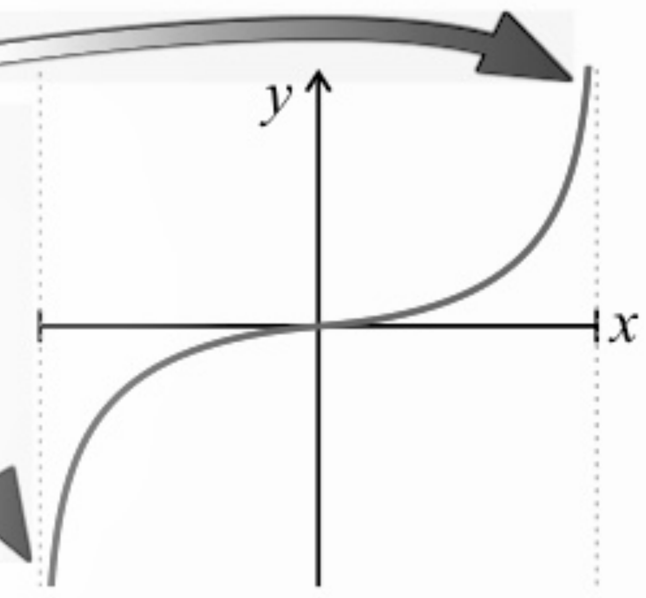
The Graph of y = tan x

The graph of $y = \tan x$ is completely different from the other two. It looks like this:



1) Unlike the graphs of sin and cos, the graph of $y = \tan x$ can give **any value** of y. The graph takes **every** y value between $-\infty$ and $+\infty$ in each **180° interval**.
2) The graph **repeats every 180°** and goes through the origin (because tan 0° = 0).
3) The graph is **not defined** at ± 90° and at ± 270°:

- These points are marked on the graph by dotted lines called **asymptotes**.
- As the graph approaches each asymptote from the **left**, it shoots **up to infinity**.
- As it approaches from the **right**, it shoots **down to minus infinity**.
- The graph **never** actually **touches** the asymptotes, although it does get **infinitely close**.

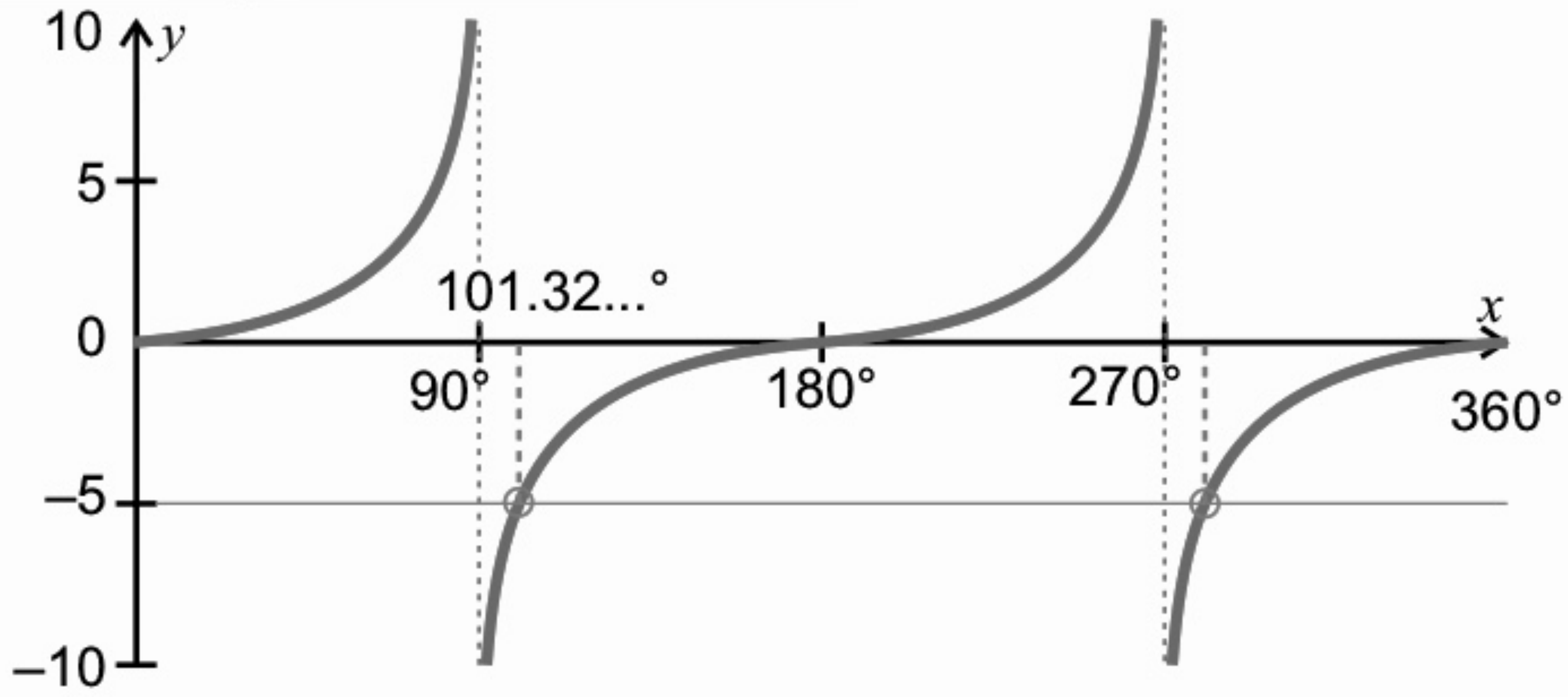


EXAMPLE: Solve $\tan x = -5$ for $0° \le x \le 360°$. Give your answer to 2 d.p.

Use your calculator to get a solution to the equation: $\tan^{-1}(-5) = -78.69...$
This is outside the interval, so add 180° to find the first solution:
$-78.69...° + 180° = 101.309...°$

Sketch the graph of $y = \tan x$ to find the solutions within the given range.

You can see from the graph that the next solution will be
$180° + 101.309...° = 281.309...°$



So the solutions are $\boldsymbol{x = 101.31°}$ **and** $\boldsymbol{x = 281.31°}$ **(2 d.p)**

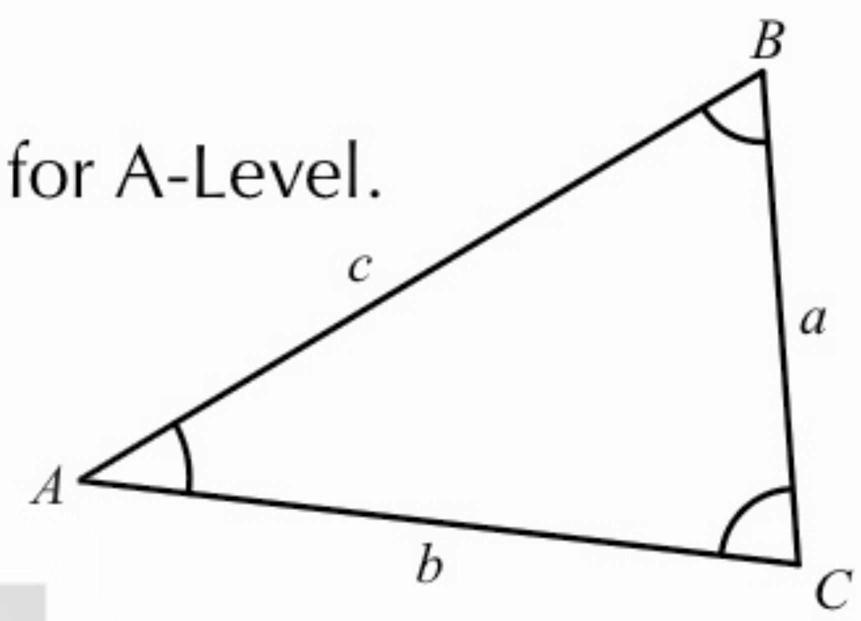
Fake tan x — it's not quite orange, but it gets infinitely close...

1) By sketching a graph, find all the solutions to the following equations in the interval $-360° \le x \le 360$. Where necessary, give your answers to 1 decimal place.
   a) $\tan x = -1$ b) $\tan x = 7$ c) $\tan x = -6$

The Sine and Cosine Rules

The *Sine* and *Cosine Rules* work for *Any Triangle*

1) To work out side lengths and angles in **any triangle** (not just right-angled ones), you'll need to use the **sine** and **cosine rules**. These came up at GCSE, but you need to be slick at using them for A-Level.
2) Be **careful** how you **label** the triangle when using the sine and cosine rules — the side **opposite** an angle should have the **same letter**. E.g. side b is opposite angle B.



The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Cosine Rule

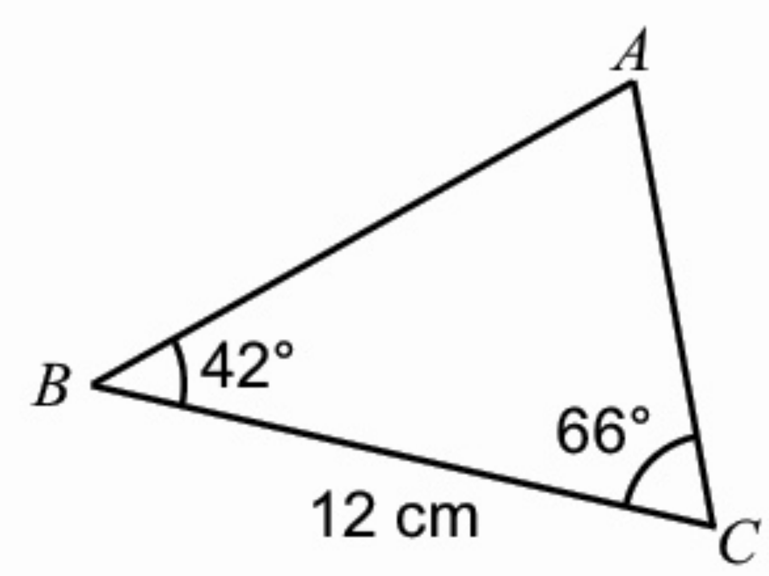
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Which **rule** you use depends on which **sides** and **angles** you know in the **triangle**.

Learn when to use the *Sine Rule*

If you have **two angles** and a **side**, you use the **sine rule** to find the lengths of the other sides. Because you have two angles, you can find the **third** by **subtracting** them from **180°**.

EXAMPLE: Find the length AC in the triangle shown.



First, find the third angle in the triangle:

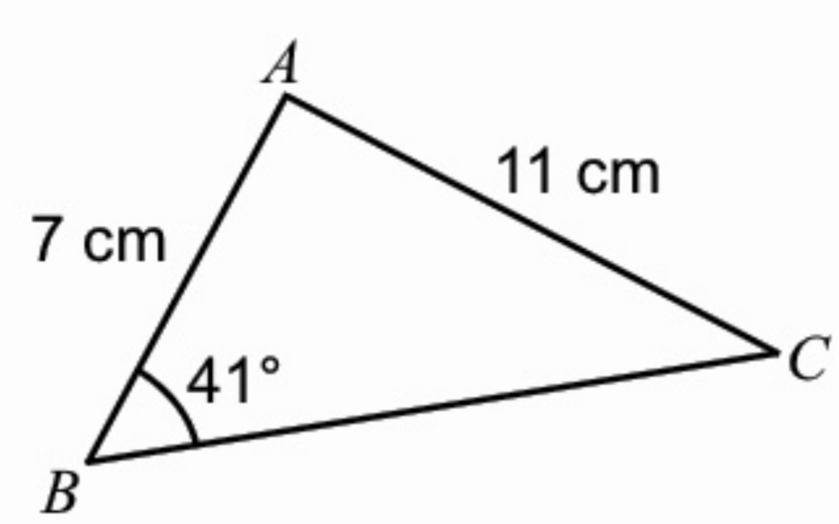
$180° - 42° - 66° = 72°$.

By the sine rule: $\dfrac{12}{\sin 72°} = \dfrac{b}{\sin 42°}$.

Rearrange to give: $b = \dfrac{12 \times \sin 42°}{\sin 72°} =$ **8.4 cm (1 d.p.)**

If you're given **two sides** and an **angle** that's **not between them**, you need to use the **sine rule**.

EXAMPLE: Find angle BAC in the triangle shown.



The known angle is not enclosed by the two known sides, so use the sine rule. Find angle C first:

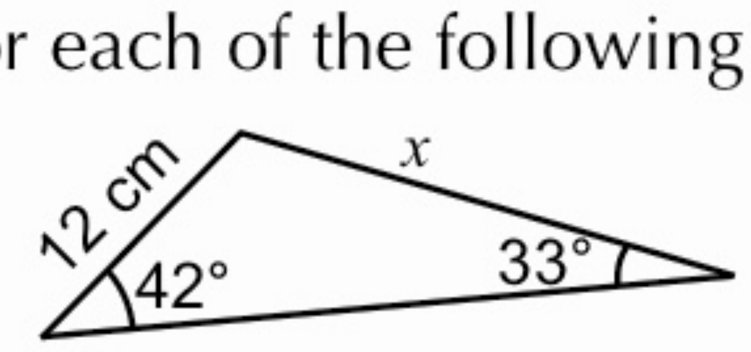
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{11}{\sin 41°} = \frac{7}{\sin C}$$

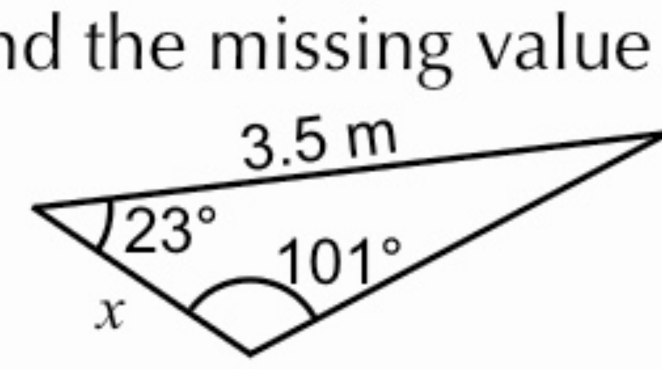
Rearrange to get $\sin C = \dfrac{7 \times \sin 41°}{11} = 0.417... \Rightarrow C = \sin^{-1}(0.417...) = 24.676...°$

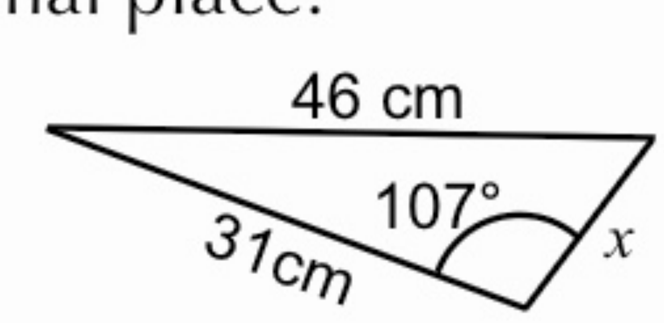
This means that angle $BAC = 180° - 41° - 24.676...° =$ **114.3° (1 d.p.)**

All these rules man — they're bringing me down...

1) For each of the following triangles, find the missing value x to 1 decimal place:

a) 

b) 

c) 

The Sine and Cosine Rules

Learn** when to use the **Cosine Rule

If you're given **two sides** and the **angle between them**,
you can use the **cosine rule** to find the length of the third side.

EXAMPLE: Two ships set sail from the same port. Ship X travels 42 miles due north. Ship Y travels 36 miles on a bearing of 035°. How far apart are the two ships?

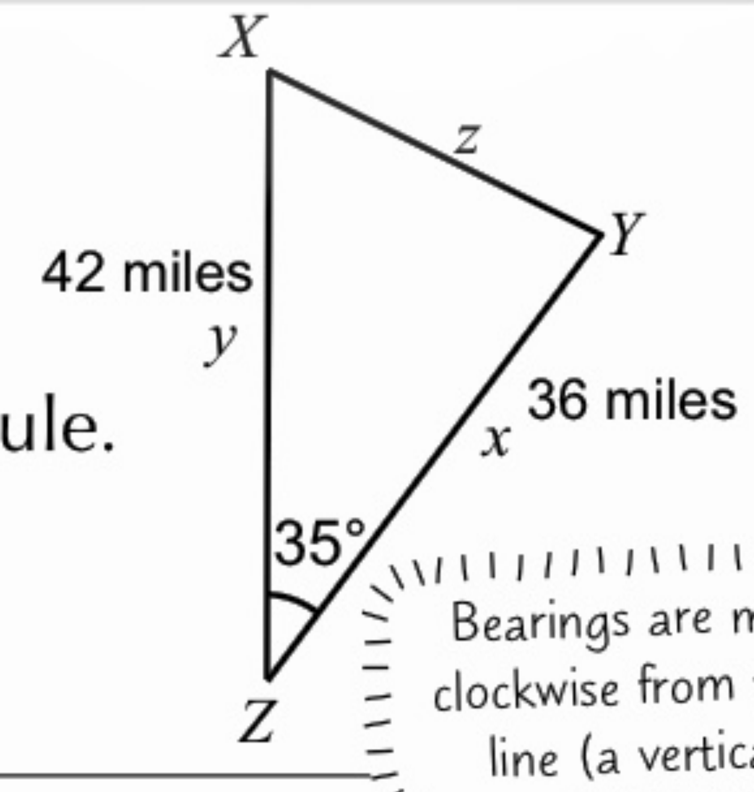
Draw a diagram and fill in the known values.

Label the sides and angles — z is the side you're trying to find.

From the diagram, you can see that the two known lengths are either side of the known angle, so you need to use the cosine rule.

$z^2 = 36^2 + 42^2 - 2 \times 36 \times 42 \times \cos 35° = 582.884....$

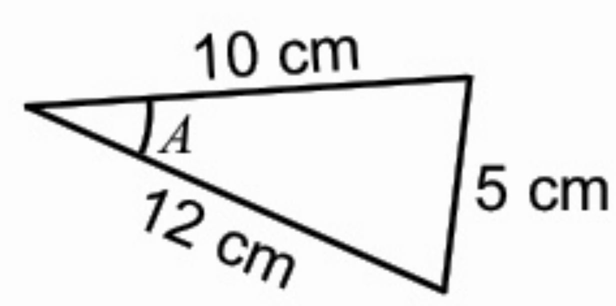
So $z = \sqrt{582.884...}$ = **24.1 miles (1 d.p.)**



Bearings are measured clockwise from the north line (a vertical line).

If you know **all three sides** of the triangle but no angles, then you need the **cosine rule**.

EXAMPLE: Find angle A in the triangle shown.



Label the sides a, b and c — remember that a is the one opposite the angle you're trying to find. So $a = 5$, $b = 10$ and $c = 12$.

Rearrange the cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Plug these into the cosine rule: $\cos A = \dfrac{10^2 + 12^2 - 5^2}{2 \times 10 \times 12} = 0.9125$

So $A = \cos^{-1} 0.9125 =$ **24.1° (1 d.p.)**

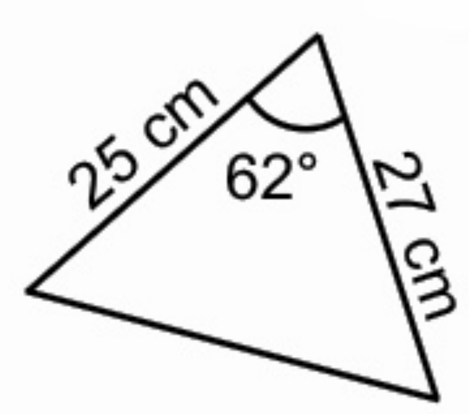
*There's also a **Formula** for the **Area** of a **Triangle***

The **area** of a triangle is given by this formula:

To use the **formula**, you need **one angle** and the **two sides enclosing the angle**. If you don't have all of these, you'll need to use the sine or cosine rule to find them.

$$\text{Area} = \frac{1}{2}ab\sin C$$

EXAMPLE: Find the area of this triangle to 1 d.p.



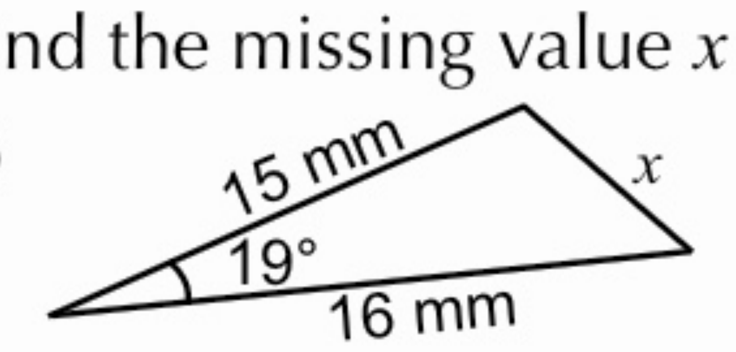
You're given an angle and two sides enclosing it, so you can plug the numbers straight into the formula.

So the area is $\frac{1}{2} \times 25 \times 27 \times \sin 62° =$ **298.0 cm² (1 d.p.)**

If you know three angles and three sides — just put your feet up...

1) Find the missing value x and the area for each of these triangles. Give your answers to 1 d.p.

a)



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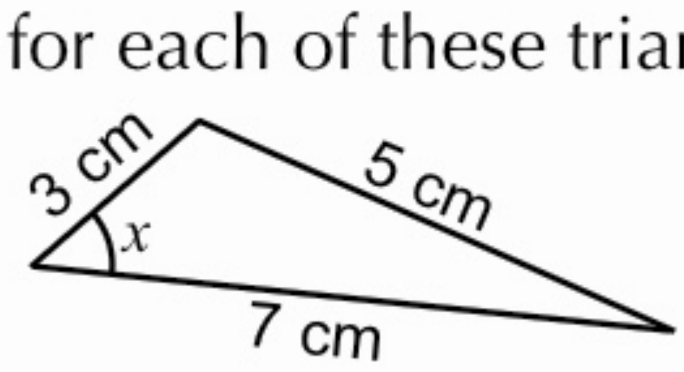
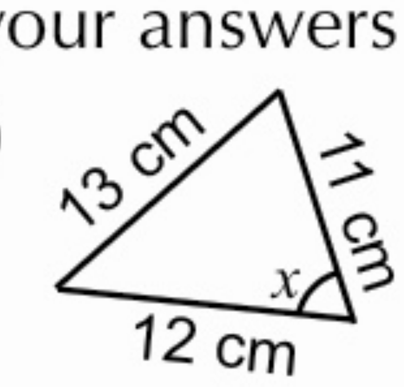


c)



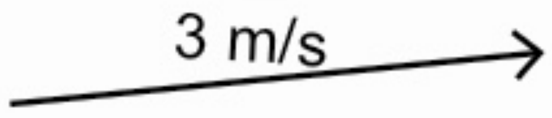
Vectors

Vectors** have a **Direction** and a **Magnitude

You will have met **vectors** at GCSE, but you're going to see a whole lot more of them at A-Level, especially in **mechanics**. They're often used for modelling things like forces.

1) **Scalars** are quantities without a direction — e.g. a **speed** of 5 m/s. They're just **numbers**.
2) **Vectors** represent a movement of a certain **magnitude (size)** in a **direction** — e.g. if two objects have a **velocity** of 5 m/s and –5 m/s, this means they are travelling at the **same speed** but in **opposite directions**.

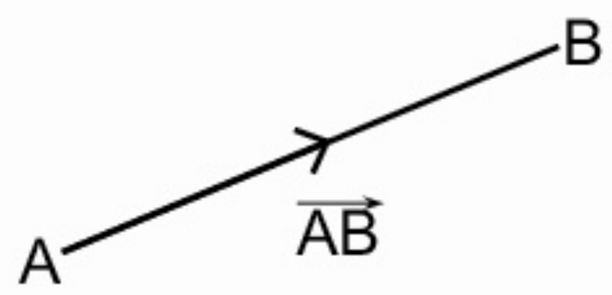
*There are **Different Ways** to **Represent Vectors***

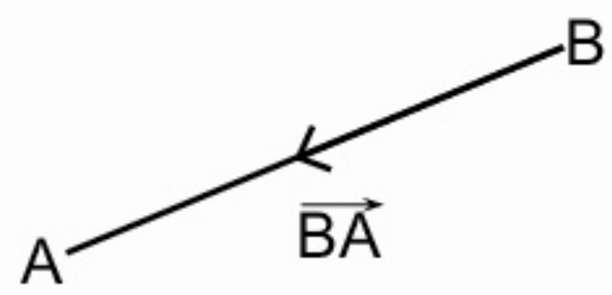
1) Vectors are **drawn** as **arrows**, like this: 

- The **direction** of the vector is shown by the **arrowhead**.
- The **size** of the vector is shown by the line's **length**.

At GCSE you may have also seen vectors written like this — $\underset{\sim}{a}$.

2) Vectors are **written** as a lowercase **bold** letter (**a**) or a lower case **underlined** letter ($\underline{a}$).
3) If the **start** and **end point** of a vector are known, it might also be written like this: $\overrightarrow{AB}$.

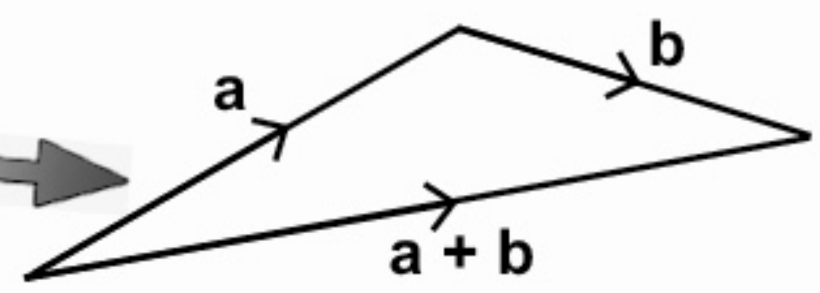




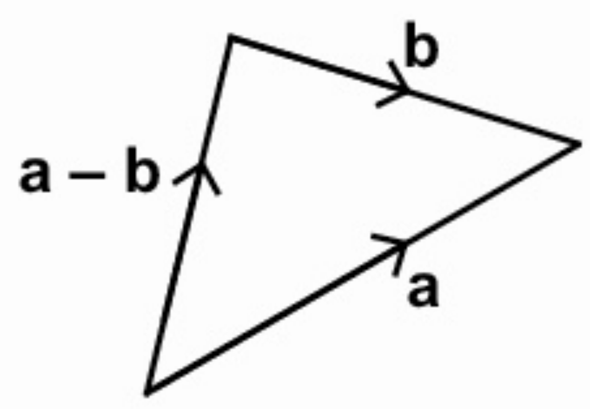
4) Vectors can also be written as column vectors — e.g. $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is a vector that goes **four units right** and **one unit up**.

Adding** Vectors Describes **Movements Between Points

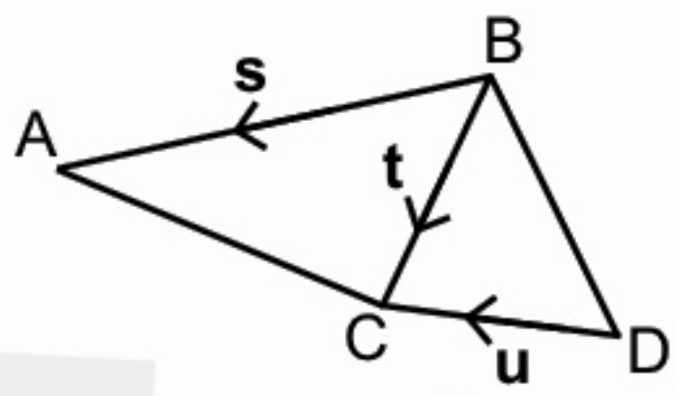
1) Adding two vectors **a** + **b** means 'go along **a** then go along **b**'.
2) To add two vectors you can draw their arrows **nose to tail**.
3) The **single** vector that goes from the **start** of the first vector in the sum to the **end** of **the last** is called the **resultant vector**.



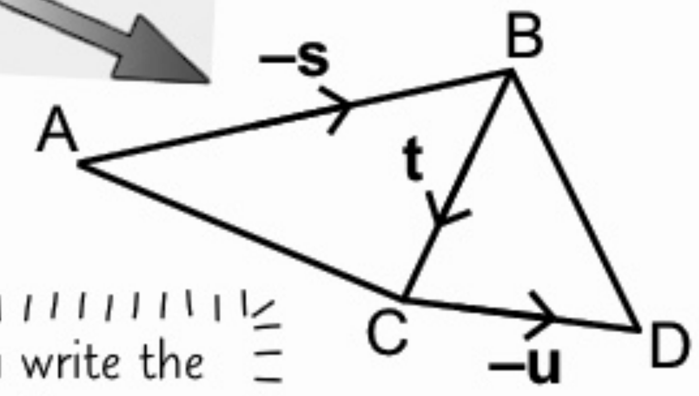
4) **Subtracting** one vector from another is a little more complicated — because –**b** is just the vector **b reversed**, you can think of **a** – **b** as **a** + (–**b**). So **a** – **b** means 'go along **a** then backwards along **b**'.



EXAMPLE: Find $\overrightarrow{AD}$ in terms of **s**, **t** and **u**.



The first thing to do is relabel the vectors so they're nose to tail from A to D like this.



Now you can see that to get from A to D, you go along **–s**, **t** and then **–u**:

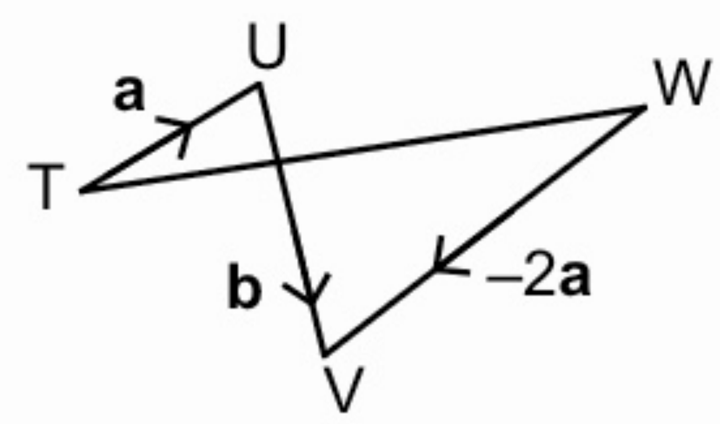
$\overrightarrow{AD}$ = –**s** + **t** + (–**u**) = **t** – **s** – **u**

The order you write the vectors in doesn't matter.

Drawing a diagram really helps with vector questions.

Woe to the scalars — and to the vector go the spoils...

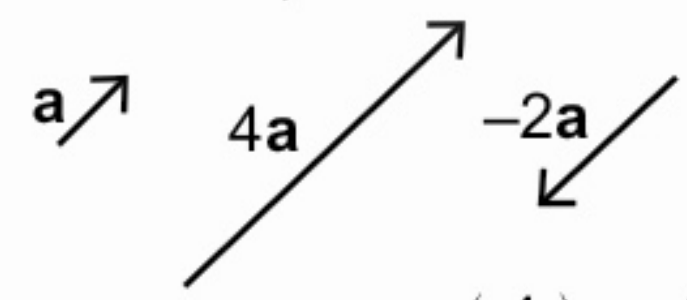
1) $\overrightarrow{AB} = 2\mathbf{s} + \mathbf{t}$ and $\overrightarrow{BC} = 2\mathbf{t} - \frac{1}{2}\mathbf{s}$. Show that $2\overrightarrow{AC} = 3(\mathbf{s} + 2\mathbf{t})$.
2) Using the diagram on the right, find $\overrightarrow{TW}$ in terms of the vectors **a** and **b**.



Vectors

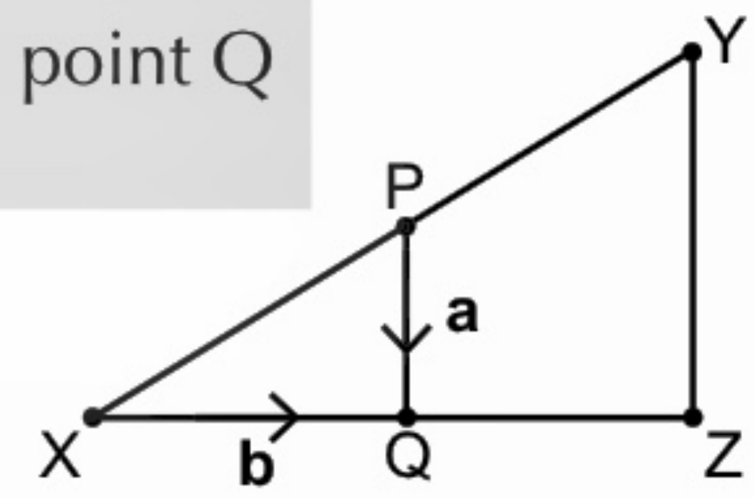
Multiplying a Vector by a Scalar Changes its Size

1) Multiplying a vector by a **positive scalar** changes the vector's **size**, but **not** its direction.
2) Multiplying a vector by a **negative scalar** changes the vector's **size** and its **direction** gets **switched**.



3) To multiply a **column vector** by a **scalar**, multiply the top and bottom numbers **separately** by the scalar, e.g. $2 \times \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \times 4 \\ 2 \times 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.
4) To show **two vectors** are **parallel**, you need to show they are **scalar multiples** of each other.

EXAMPLE: $\overrightarrow{XQ} = \mathbf{b}$ and $\overrightarrow{PQ} = \mathbf{a}$. Point P lies halfway along $\overrightarrow{XY}$ and point Q lies halfway along $\overrightarrow{XZ}$. Show that $\overrightarrow{YZ}$ is parallel to $\overrightarrow{PQ}$.

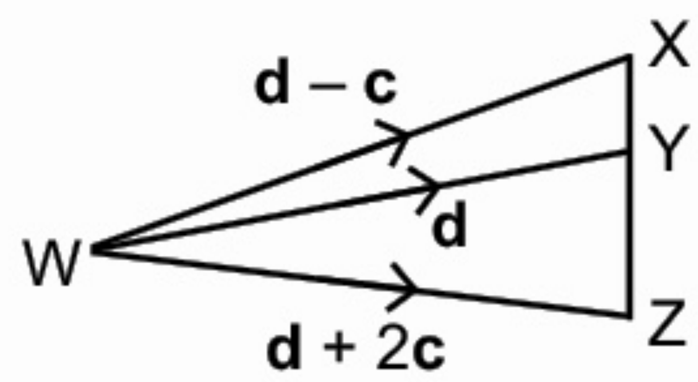


As Q is the midpoint of $\overrightarrow{XZ}$, $\overrightarrow{XZ} = 2\overrightarrow{XQ} = 2\mathbf{b}$.
$\overrightarrow{XP} = \mathbf{b} - \mathbf{a}$. P is the midpoint of $\overrightarrow{XY}$, so $\overrightarrow{XY} = 2\overrightarrow{XP} = 2(\mathbf{b} - \mathbf{a})$.
$\overrightarrow{YZ} = \overrightarrow{YX} + \overrightarrow{XZ} = -\overrightarrow{XY} + \overrightarrow{XZ} = -2(\mathbf{b} - \mathbf{a}) + 2\mathbf{b} = 2\mathbf{a}$
So $\overrightarrow{YZ} = 2\overrightarrow{PQ} \Rightarrow \overrightarrow{YZ}$ is a scalar multiple of $\overrightarrow{PQ}$, so these vectors are **parallel**.

You can use Vectors to Show Three Points are on a Line

1) If **three or more points** all lie on a **single straight line**, they are **collinear**.
2) To show three points A, B and C are collinear, you need to show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.
3) So you need to show that the vectors are **scalar multiples** of one another.

EXAMPLE: Show that X, Y and Z are collinear.



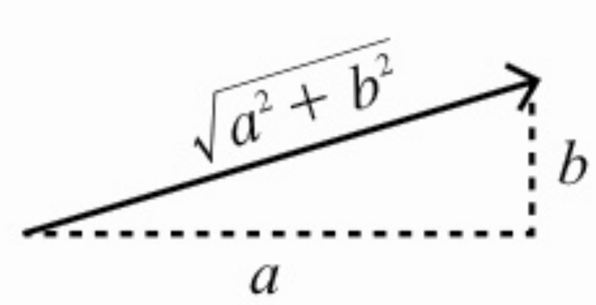
First, find $\overrightarrow{XY}$ and $\overrightarrow{YZ}$:
$\overrightarrow{XY} = -(\mathbf{d} - \mathbf{c}) + \mathbf{d} = \mathbf{c}$ and $\overrightarrow{YZ} = -\mathbf{d} + \mathbf{d} + 2\mathbf{c} = 2\mathbf{c}$.
$\boldsymbol{\overrightarrow{YZ} = 2\overrightarrow{XY}}$**, which means** $\boldsymbol{\overrightarrow{XY}}$ **and** $\boldsymbol{\overrightarrow{YZ}}$ **are parallel, so X, Y and Z are collinear.**

A Vector's Magnitude is its Length

The magnitude of a vector **a** is written |**a**|. The magnitude of $\overrightarrow{AB}$ is written $|\overrightarrow{AB}|$

At **A-Level** you'll have to calculate the **magnitude of a vector**.
To do this, you need to work out the **distance** between the **start point** and the **end point**.
You do this using Pythagoras' theorem (see page 35).

Magnitude of vector $\begin{pmatrix} a \\ b \end{pmatrix} = \sqrt{a^2 + b^2}$



Getting from point A to point C has never been so exciting...

1) Show that the vectors $\mathbf{v} = 4\mathbf{a} + 6\mathbf{b}$ and $\mathbf{u} = 6\mathbf{a} + 9\mathbf{b}$ are parallel.
2) Using the diagram on the right, show that W, X and Y are collinear.
3) Calculate the magnitude of the following vectors: a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$