# Factorising Quadratics

## Quadratics have a Specific Form

**Quadratics** are of the form $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are **numbers**.
A quadratic equation can't have any other powers of $x$, like $x^3$ or $x^{\frac{1}{2}}$.
You'll need to be really confident with quadratics for A-level — they crop up **everywhere**, including in later work on **polynomials** and **graphs** of functions.

'$a$' and '$b$' are the coefficients of $x^2$ and $x$. '$c$' is called the constant term.

## Factorising Quadratics is Easier when a = 1

1) Normally, you'll factorise your **quadratic** by splitting it up into **two** sets of **brackets** — $x^2 + bx + c = (x + p)(x + q)$.
2) As well as factorising a quadratic, you might be asked to **solve** the equation. This just means finding the **values** of $x$ that make each **bracket** equal 0 (see example below).
3) Factorising quadratics is easier when the coefficient of $x^2$ is 1.

This first step might involve multiplying or dividing. E.g. dividing $3x^2 + 6x + 3 = 0$ by 3 gives $x^2 + 2x + 1 = 0$.

1) Rearrange your equation into the **standard format**: $x^2 + bx + c = 0$
2) Write down the **two brackets** with the $x$'s in: $(x \quad)(x \quad) = 0$
3) Now you need to find 2 numbers that **multiply** together to give the value '$c$' (the constant) but also **add/subtract** to give the value '$b$' (the coefficient of $x$).
4) Put in the **+/–** signs and make sure they give the right numbers.
5) **Check** this works by **expanding the brackets** to make sure this gives you the original equation.
6) To solve the equation, set each **bracket equal to 0** and solve for $x$.

For more on expanding brackets, see page 10.

**EXAMPLE:** Solve $x^2 + 7x + 12 = 0$.

This is already in the standard format, with $b = 7$ and $c = 12$.
Write out the initial brackets: $(x \quad)(x \quad) = 0$.
You need to find a pair of numbers that multiply to give $c$ (= 12) and add together to give $b$ (= 7): $3 \times 4 = 12$ and $3 + 4 = 7$ — perfect.
Write these in the brackets, and then fill in the +/– to make 3 and 4 add to give 7: $(x + 3)(x + 4) = 0$.
Double check that this works by multiplying out the brackets
$(x + 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$ ✔
Set the brackets equal to 0 to solve: $(x + 3) = 0$ so $\boldsymbol{x = -3}$ and $(x + 4) = 0$ so $\boldsymbol{x = -4}$

If $c$ is positive, then both signs are the same. If $c$ is negative, then the signs will be opposites.

## What do you call a group of equations? A squadratic...

1) Solve the following quadratic equations:
   a) $x^2 + 3x + 2 = 0$ b) $x^2 + 8x + 7 = 0$ c) $x^2 = 2(7x - 20)$ d) $x^2 - x = 6$
   e) $7x = x^2 + 10$ f) $\frac{x^2}{2} + 2x - 6 = 0$ g) $x - 4 - \frac{12}{x} = 0$ h) $2x^2 - 2x - 4 = 0$
2) Factorise $x^2 - 2xz + z^2$.
3) In an experiment, the temperature $T$ °C is modelled by $T = -m^2 + 13m - 30$, where $m$ is the time in minutes after the start of the experiment. Find both times at which the temperature is 0 °C.

# Factorising Quadratics

## Factorising a **Quadratic** is **Trickier** when $a \neq 1$

When $a \neq 1$, the basic method is still the same as before, but it's just a bit more **fiddly**.

1) When you write down the two brackets, they'll now be of the form $(nx\quad)(mx\quad)$, where $n$ and $m$ are two numbers that **multiply** to give $a$.
2) Now you're looking for two numbers that **multiply** to give $c$, but which also give you $bx$ when you **multiply** them by $nx$ and $mx$, and then **add / subtract** them.
3) Put those two numbers in the brackets, and then choose their **signs** to make it work.

**EXAMPLE:** Solve $2x^2 + 7x = -6$.

First rearrange the equation into the standard format: $2x^2 + 7x + 6 = 0$

Now, write out the two brackets. The $x$ bits need to multiply together to be $2x^2$ so the brackets will be of the form $(2x\quad)(x\quad)$.

Now, find pairs of numbers that multiply to give $c$ (= 6).
Number pairs $1 \times 6$ and $2 \times 3$ both work.

Try these number pairs out in the brackets until you find a pair that gives $7x$.
You'll need to try each number in 2 positions because the brackets are different.
$(2x\quad 1)(x\quad 6)$ multiplies to give $12x$ and $x$ which add/subtract to give $13x$ or $11x$. ✗
$(2x\quad 6)(x\quad 1)$ multiplies to give $2x$ and $6x$ which add/subtract to give $8x$ or $4x$. ✗
$(2x\quad 2)(x\quad 3)$ multiplies to give $6x$ and $2x$ which add/subtract to give $8x$ or $4x$. ✗
$(2x\quad 3)(x\quad 2)$ multiplies to give $4x$ and $3x$, which add/subtract to give $x$ or $7x$. ✔

This means you can fill in the +/– signs so that 4 and 3 add to give 7: $(2x + 3)(x + 2)$.

Check that this works by multiplying it out:
$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6 = 2x^2 + 7x + 6$ ✔

Set the brackets equal to zero and solve: $(2x + 3) = 0$, so $\boldsymbol{x = -\frac{3}{2}}$ and $(x + 2) = 0$, so $\boldsymbol{x = -2}$

**EXAMPLE:** Factorise $3x^2 + 4x - 7$.

The brackets will be $(3x\quad)(x\quad)$.

The only possible number pairing is 1 and 7, so try each number in both positions:
$(3x\quad 1)(x\quad 7)$ multiplies to give $21x$ and $x$ which add/subtract to give $22x$ or $20x$.
$(3x\quad 7)(x\quad 1)$ multiplies to give $3x$ and $7x$ which add/subtract to give $10x$ or $4x$. ✔

Finally work out the signs for each set of brackets: $3x^2 + 4x - 7 = \boldsymbol{(3x + 7)(x - 1)}$.

Don't forget to multiply out the brackets to check your answer is correct.

## Oh drat... Double drat.... Triple drat... Quad drat it...

1) Solve the following quadratic equations:
   a) $2x^2 + 9x + 9 = 0$  b) $5x^2 + 13x + 6 = 0$  c) $2x^2 = x + 10$  d) $3x + \frac{21}{x} = 16$
2) The equation $h = -4t^2 + 2t + 2$ models the height $h$ m a paper plane is from the ground, where $t$ is time after the plane is thrown in seconds. At what time $t$ does the plane hit the ground?

# The Quadratic Formula

## *The **Quadratic Formula** Gives Solutions to **Quadratic Equations***

The solutions to **any** quadratic equation in the form $ax^2 + bx + c = 0$ are given by this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It means you can solve quadratics that are too hard to factorise.

1) To find $x$, you just need to put the values of $a$, $b$ and $c$ into the formula.
2) The $\pm$ sign means you can end up with two solutions — in the final stage you replace it with '+' and then '–'.

**EXAMPLE:** Solve $8x^2 + 3x - 3 = 0$, giving your answers to 3 decimal places.

$a = 8$, $b = 3$ and $c = -3$. So put these values into the formula:

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 8 \times -3}}{2 \times 8} = \frac{-3 \pm \sqrt{9 + 96}}{16} = \frac{-3 \pm \sqrt{105}}{16}$$

So $x = \frac{-3 + \sqrt{105}}{16}$ and $x = \frac{-3 - \sqrt{105}}{16}$ are the solutions — you should leave your answers in this form if you're asked for exact solutions.

Put these into your calculator to get: $\boldsymbol{x = 0.453}$ and $\boldsymbol{x = -0.828}$

The question mentions decimal places, which suggests you'll need the formula to solve the quadratic.

At A-Level, you might see quadratics which result in negative square roots in the quadratic formula (see below).

## *The **Discriminant** tells you about the **Roots** of a **Quadratic***

The **discriminant** is the bit of the quadratic formula that's **inside** the **square root**: $\boldsymbol{b^2 - 4ac}$

It's used in A-Level Maths to find **how many roots** a quadratic has.
The roots are the **values of $x$** where the **graph** of the quadratic **crosses the $x$-axis**.

1) If the discriminant is **positive** ($b^2 - 4ac > 0$), this means the quadratic has **two real roots**. The formula will give **two different real number values** of $x$ — one from adding and one from subtracting the $\sqrt{b^2 - 4ac}$ bit.
2) If the discriminant is **zero**, the quadratic only has **one root** because adding or subtracting 0 will give the same value of $x$.
3) If the discriminant is **negative** ($b^2 - 4ac < 0$) this means the quadratic has **no (real) roots** because the square root of a negative number is **not** a real number (see page 6).

**EXAMPLE:** Find the discriminant of $4x^2 + 3x + 1$.
How many real roots does $4x^2 + 3x + 1 = 0$ have?

$a = 4$, $b = 3$ and $c = 1$, so the discriminant is $b^2 - 4ac = 3^2 - 4 \times 4 \times 1 = 9 - 16 = -7$

The discriminant is $< 0$, so $4x^2 + 3x + 1 = 0$ has **no real roots**.

## *Due to the poor quality of previous jokes, this one has been removed...*

1) Solve the following quadratic equations. Give your answers to 3 d.p.

   a) $4x^2 - 7x + 1 = 0$  b) $6x^2 + x = 4$  c) $-2x^2 = 3x - 4$

2) Find the discriminant of $5x^2 + 7x + 3$. How many real roots does $5x^2 + 7x + 3 = 0$ have?

# Completing the Square

## *Completing The Square when a = 1*

**Completing the square** means rewriting a **quadratic** as a **squared bracket** plus or minus a **number**. It's used at A-Level for **sketching graphs** (see next page) and **solving the equations** of quadratics (see below) and it's also used with the **equations of circles**. This is the method for when $a$ (the coefficient of $x^2$) is 1:

1) Start by getting your quadratic into the **standard format**: $ax^2 + bx + c$.
2) Write out the initial **squared bracket** $(x + \frac{b}{2})^2$ — so just **divide** the coefficient of $x$ by **2**.
3) Now, **multiply out** the squared bracket and **compare** what you get with the **original** quadratic to see what you need to add or subtract.
4) Finally, add or subtract the **adjusting number** to make it **match** the original.

**EXAMPLE:** Rewrite $x^2 + 12x + 16$ in the form $(x + m)^2 + n$.

To find the adjusting number, you can just take the number term you get from expanding the bracket away from '$c$'.

This quadratic's already in the standard format, so go straight to step 2 and write out the initial squared bracket: $(x + 6)^2$

Multiply this out and compare with the original: $(x + 6)^2 = x^2 + 12x + 36$

This has a 36 at the end, but you need a 16 at the end, so adjust by $16 - 36 = \mathbf{-20}$

$(x + 6)^2 - 20 = x^2 + 12x + 36 - 20 = x^2 + 12x + 16$ ✔

So the completed square is $\mathbf{(x + 6)^2 - 20}$

## *The Completed Square can help you Solve a Quadratic*

Once you've **completed the square**, you can **solve** a quadratic. All you have to do is **rearrange** the completed square to make $x$ the **subject** by following these three steps:

1) Move the **adjusting number** to the **other side** of the equation.
2) **Square root** both sides — **remember** the '±'...
3) Rearrange to get $x$ on its **own**.

**EXAMPLE:** Given that $x^2 - 14x + 36 = (x - 7)^2 - 13$, solve $x^2 - 14x + 36 = 0$.

Set the completed square equal to zero to solve $(x - 7)^2 - 13 = 0$

Remember, when you take a square root you get two values — one positive and one negative.

Now, move the adjusting number to the RHS: $(x - 7)^2 = 13$

Next, square root both sides to get: $x - 7 = \pm\sqrt{13}$

Finally, add seven to each side to get $x$ on its own: $\mathbf{x = 7+\sqrt{13}}$ and $\mathbf{x = 7-\sqrt{13}}$

## *Complete Trafalgar Square — Nelson's Column + 4 plinths + $5^{10}$ pigeons...*

1) Write each of the following in the form $(x + m)^2 + n$:

a) $x^2 + 4x + 1$ b) $x^2 - 12x + 5$ c) $x^2 - 20x - 10$ d) $x^2 + 7x - 3$

2) Write $x^2 + 6x - 8$ in the form $(x + m)^2 + n$. Hence solve $x^2 + 6x - 8 = 0$.

# Completing the Square

## You can *Sketch* the *Graph* Using the *Completed Square*

The completed square form of a quadratic gives you information about its graph.
When the coefficient of $x^2$ is **positive**:

If the adjusting number is 0, the equation has one root.

- The graph is **u-shaped** with a **minimum point** (see p.37).
- The **adjusting number** tells you the ***y*-value** of this minimum point. This is because the **smallest value** the quadratic can take occurs when the bit in the brackets is **zero** (the bracket is squared so can never give a negative value).
- If the **adjusting number** is **positive** then the graph's **minimum point** is **above** the $x$-axis. So the graph **never** crosses the $x$-axis — it has **no real roots**.

When the coefficient of $x^2$ is **negative** (there are examples where $a$ isn't 1 on the next page):

- The graph is **n-shaped** with a **maximum point** (see p.37).
- The **maximum point** occurs when the bit in brackets is **zero** and the **adjusting number** tells you the ***y*-value** of this maximum point.
- If the **adjusting number** is **negative** then the graph's **maximum point** is **below** the ***x*-axis**. So the graph **never** crosses the $x$-axis, meaning it has **no real roots**.

As you saw on the last page, the completed square can also be used to find the solutions of the quadratic. So it can tell you where the graph **crosses the *x*-axis** (if it crosses the $x$-axis at all).

**EXAMPLE:** Sketch the graph of $y = x^2 - 14x + 36$.

There's lots more about sketching quadratics on p.37.

You know from the example on the previous page that $x^2 - 14x + 36 = (x - 7)^2 - 13$.

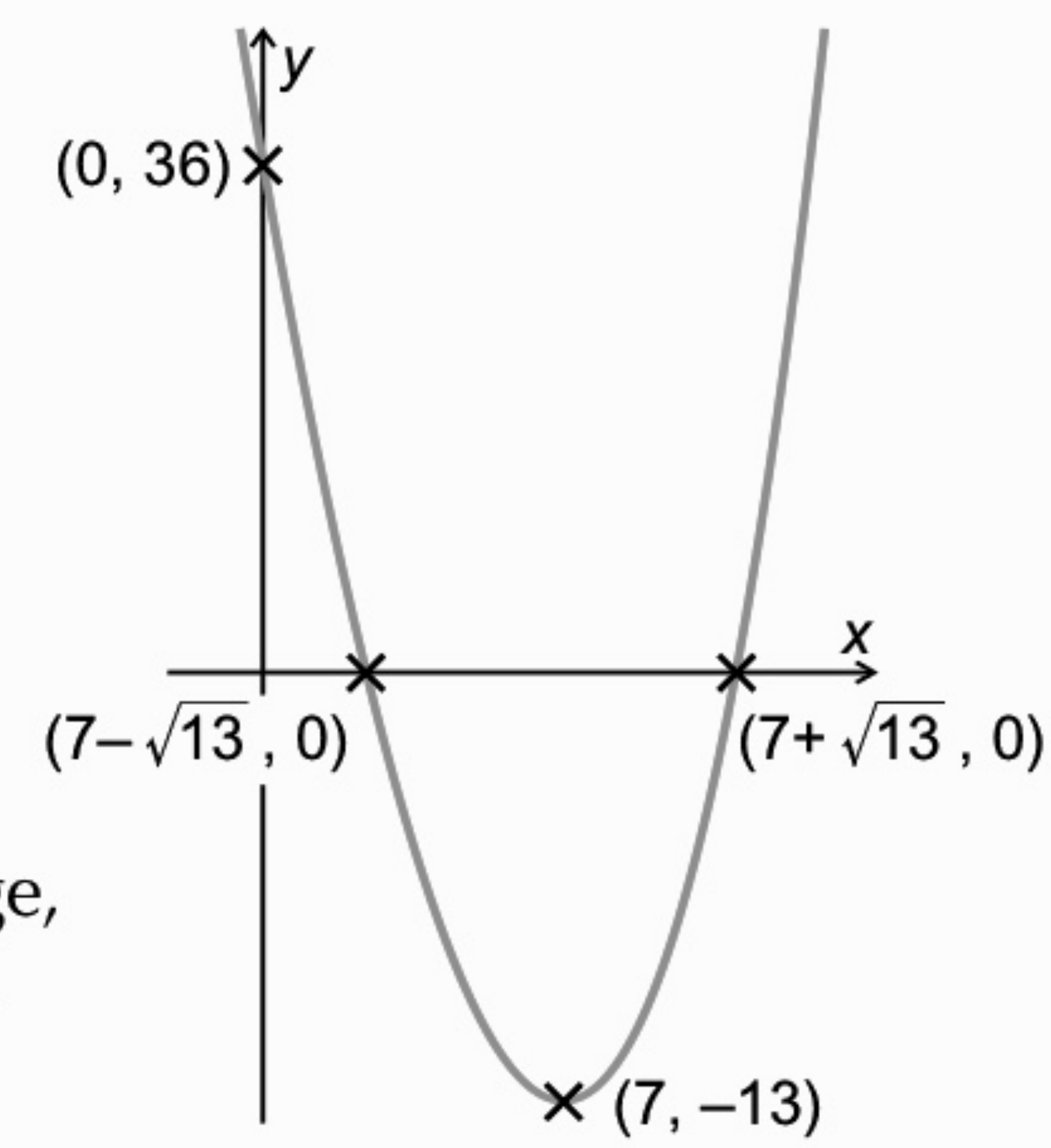
The coefficient of $x^2$ is positive, so it is u-shaped and has a minimum point.

The graph takes its minimum value when $(x - 7)^2 = 0$, so when $x = 7$.
The minimum $y$-value is the adjusting number (–13).

So the minimum point of the graph is at (7, –13).

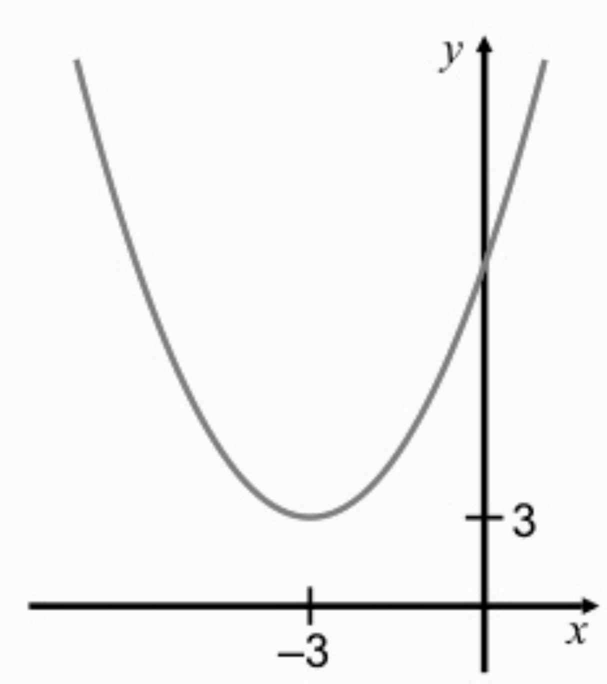
The graph crosses the $x$-axis at the solutions to the quadratic equation. From the example on the previous page, you know these solutions are $x = 7+\sqrt{13}$ and $x = 7 - \sqrt{13}$.

To find where the graph crosses the $y$-axis, let $x = 0$:
$0^2 - (14 \times 0) + 36 = 36$, so it crosses at (0, 36).




## Complete Times Square — Billboards + billboards + billboards + billboards + ...

1) a) Write $x^2 + 16x + 3$ in the form $(x + a)^2 + b$.
   b) Hence solve $x^2 + 16x + 3 = 0$.
   c) Use this information to sketch the graph of $y = x^2 + 16x + 3$.
2) The graph on the right shows the graph of $y = (x + p)^2 + q$, where $p$ and $q$ are integers.
   a) Use the sketch to find the values of $p$ and $q$.
   b) Find the $y$-intercept of the graph.



# Completing the Square

## *Completing The Square where a ≠ 1*

If $a$ **isn't 1**, you need to take a **factor** of $a$ from the $x^2$ and $x$ terms before you can complete the square.

This works for negative or fractional values of '$a$' as well.

**EXAMPLE:** Write $3x^2 - 8x = -4$ in the form $p(x + q)^2 + r = 0$.

Start by putting the quadratic into the standard format: $3x^2 - 8x + 4 = 0$.

Take out a factor of 3 from the $x^2$ and $x$ terms: $3(x^2 - \frac{8}{3}x) + 4 = 0$.

Now, write out the initial bracket: $3(x - \frac{8}{6})^2 = 3(x - \frac{4}{3})^2$

Multiply out the bracket and compare to the original: $3(x - \frac{4}{3})^2 = 3x^2 - 8x + \frac{16}{3}$

The adjusting number is $4 - \frac{16}{3} = -\frac{4}{3}$:

$3(x - \frac{4}{3})^2 - \frac{4}{3} = 3x^2 - 8x + \frac{16}{3} - \frac{4}{3} = 3x^2 - 8x + 4$ ✔

So the completed square is $\mathbf{3(x - \frac{4}{3})^2 - \frac{4}{3}}$

## *Completing the Square is useful for Circle Equations*

At A-Level, you'll sometimes need to rearrange the **equation of a circle** so you can find its **centre** and **radius**. There's lots more about the equations of circles on page 39. With circles you might need to complete the square **twice** — once for $x$ and once for $y$.

**EXAMPLE:** Write $x^2 + y^2 - 6x + 14y + 22 = 0$ in the form $(x - m)^2 + (y - n)^2 = k$.

Start by rearranging the equation so the $x$'s and $y$'s are grouped together:
$x^2 - 6x + y^2 + 14y + 22 = 0$

Now you need to complete the square for both $x$ and $y$.
Completing the square for $x^2 - 6x$ gives $(x - 3)^2 - 9$,
so the equation so far is: $(x - 3)^2 - 9 + y^2 + 14y + 22 = 0$

Completing the square for $y^2 + 14y$ gives: $(y + 7)^2 - 49$,
so the equation is: $(x - 3)^2 - 9 + (y + 7)^2 - 49 + 22 = 0$

Finally, rearrange into the form $(x - m)^2 + (y - n)^2 = k$:
Combine all the number terms together: $(x - 3)^2 + (y + 7)^2 - 36 = 0$
Put the number term on the other side of the equation: $\mathbf{(x - 3)^2 + (y + 7)^2 = 36}$

## *Complete Red Square — Kremlin + fine this joke's getting old now...*

1) Complete the square for each of these quadratics:
   a) $2x^2 - 3x - 10$  b) $-x^2 - 3x - 1$  c) $5x^2 - x - 3$  d) $-\frac{1}{2}x^2 - 3x - 7$
2) Write $x^2 + y^2 - 2x + 6y - 15 = 0$ in the form $(x - m)^2 + (y - n)^2 = k$.

# Algebraic Fractions

## *Simplify* Algebraic Fractions by *Cancelling*

To simplify an **algebraic fraction**, you need to find **common factors** in the numerator and denominator. Then you can **divide** the numerator and denominator by the common factors to cancel them. Some fractions won't cancel straight away — you'll need to **factorise** first.

**EXAMPLE: a)** Simplify $\frac{36ab^5}{27a^2b^3}$.

The common factors in the numerator and denominator are 9, $a$ and $b^3$.
Cancel down by dividing the numerator and denominator by each one: $\frac{{}^4\cancel{36}\cancel{a}\cancel{b^5}^{b^2}}{{}_3\cancel{27}\cancel{a^2}_a\cancel{b^3}} = \mathbf{\frac{4b^2}{3a}}$

**b)** Simplify $\frac{8x^3 - 4x^2}{6x^2 + 5x - 4}$.

Factorising the expressions in the numerator and denominator gives: $\frac{4x^2(2x-1)}{(2x-1)(3x+4)}$

Then cancelling $(2x - 1)$ gives: $\frac{4x^2\cancel{(2x-1)}}{\cancel{(2x-1)}(3x+4)} = \mathbf{\frac{4x^2}{3x+4}}$

Take a look back at p.16-17 to recap factorising quadratics.

## *Multiplying* and *Dividing* Algebraic Fractions

To **multiply** algebraic fractions, multiply the numerators and denominators **separately**.
To **divide** algebraic fractions, turn the second fraction **upside down** and then **multiply**.

**EXAMPLE: a)** Simplify $\frac{x+3}{2x} \times \frac{x^2}{(x+3)^2}$.

Always do as much cancelling down as you can before multiplying.

Cancel any common factors first: $\frac{\cancel{x+3}}{2\cancel{x}} \times \frac{\cancel{x^2}^{x}}{(x+3)^{\cancel{2}}}$

Then multiply the numerator and denominator separately: $\mathbf{\frac{x}{2(x+3)}}$

You've seen this method with numbers on page 7.

**b)** Simplify $\frac{x^2-16}{8} \div \frac{(x-4)}{2x}$.

You can factorise $x^2 - 16$ using the rule for a difference of two squares — see p.11.

Start by flipping the second fraction upside down to give: $\frac{x^2-16}{8} \times \frac{2x}{(x-4)}$

Factorise and cancel the common factors $(x - 4)$ and 2: $\frac{\cancel{(x-4)}(x+4)}{\cancel{8}_4} \times \frac{\cancel{2}x}{\cancel{(x-4)}}$

Then multiply to get the answer: $\mathbf{\frac{x(x+4)}{4}}$

## *Flipping, factorising, multiplying — so much to do and so little time...*

1) Simplify these fractions:

a) $\frac{3x^2}{7x}$ b) $\frac{8x+16}{2x-4}$ c) $\frac{x^2-25}{5(x+5)}$ d) $\frac{3x^2+16x-12}{2x^2+13x+6}$ e) $\frac{x^3+2x^2+x}{x^2-3x-4}$

2) Simplify these expressions:

a) $\frac{x+3}{x^2} \times \frac{x}{4}$ b) $\frac{3x+9}{4} \times \frac{x}{3(x+3)}$ c) $\frac{10x}{x^2-9} \div \frac{2x+14}{x-3}$ d) $\frac{x^2+5x+6}{9} \div \frac{x^2-4x-21}{6x-42}$

# Algebraic Fractions

## Adding and Subtracting Algebraic Fractions is a Little Tougher

Before you add or subtract algebraic fractions, take a look at the **denominators**. If they're all the **same**, they have a **common denominator**, so you can just add up the numerators.

$$\frac{2}{x}+\frac{5a}{x}+\frac{2a^2}{x}=\frac{2+5a+2a^2}{x}$$

But when the denominators are **different**, you'll need to:

1) Find a **common denominator** — this should be the **lowest common multiple** of all the denominators.
2) **Rewrite** each fraction with the common denominator by **multiplying** the top and bottom by the **same thing**.
3) Then make into one fraction by **adding** or **subtracting** the **numerators**.

These skills are really useful — algebraic fractions pop up in both years of A-Level.

**EXAMPLE: a)** Simplify $\frac{3x+y}{2}-\frac{x-2y}{7}$.

The LCM of 2 and 7 is 14, so rewrite each fraction with a denominator of 14. Multiply the top and bottom of the first fraction by 7, and the top and bottom of the second fraction by 2:

$$\frac{3x+y}{2}-\frac{x-2y}{7}=\frac{7(3x+y)}{14}-\frac{2(x-2y)}{14}=\frac{21x+7y-2x+4y}{14}=\mathbf{\frac{19x+11y}{14}}$$

**b)** Simplify $\frac{1}{x}+\frac{3}{x+1}$.

Multiply $x$ and $(x+1)$ together to get the common denominator $x(x+1)$. Rewrite each fraction with a denominator of $x(x+1)$ to get:

$$\frac{1}{x}+\frac{3}{x+1}=\frac{1(x+1)}{x(x+1)}+\frac{3x}{x(x+1)}=\frac{x+1+3x}{x(x+1)}=\mathbf{\frac{4x+1}{x(x+1)}}$$

You can use exactly the same method to add and subtract **more than two** fractions.

**EXAMPLE:** Simplify $\frac{1}{4}+\frac{1}{2x}-\frac{1}{4x}$.

The LCM of the three denominators is $4x$ — so this is the simplest common denominator.

Rewriting each fraction over $4x$ gives: $\frac{x}{4x}+\frac{2}{4x}-\frac{1}{4x}=\mathbf{\frac{x+1}{4x}}$

## *Adding and subtracting three fractions? Now you're just being awkward...*

1) Express these as a single fraction:

a) $3+\frac{2}{x}$ b) $\frac{a}{b}-\frac{2a}{3b}$ c) $\frac{1}{x+1}-\frac{3}{x}$ d) $\frac{4}{x+2}+\frac{3}{x^2}$

e) $\frac{1}{x+1}-\frac{1}{x(x+1)}+\frac{1}{x}$ f) $\frac{5x}{7}+\frac{x+3}{2x}$ g) $\frac{x-1}{x+8}-\frac{3}{x+1}$ h) $1+\frac{4}{x-4}-\frac{2}{x+2}$

# Inequalities

## *Solving **Inequalities** is Just Like Solving **Equations**...*

...whatever you do to one side, you also do to the other side. And follow this important rule:

If you **multiply** or **divide** by a **negative number**, **flip** the inequality sign.

Inequalities in the form $ax + b > cx + d$ are known as linear inequalities.

**EXAMPLE: a)** Solve $8x < 5x - 3$.

Subtract $5x$ from both sides:
$8x - 5x < 5x - 3 - 5x$
$3x < -3$
Then divide by 3:
$3x \div 3 < -3 \div 3$
$\boldsymbol{x < -1}$

**b)** Solve $3(6 - x) \geq 12$.

Multiply out the brackets first: $18 - 3x \geq 12$
Subtract 18 from both sides:
$18 - 3x - 18 \geq 12 - 18$
$-3x \geq -6$
Divide by $-3$:
$-3x \div -3 \leq -6 \div -3$
$\boldsymbol{x \leq 2}$

You've divided by a negative number, so flip the sign.

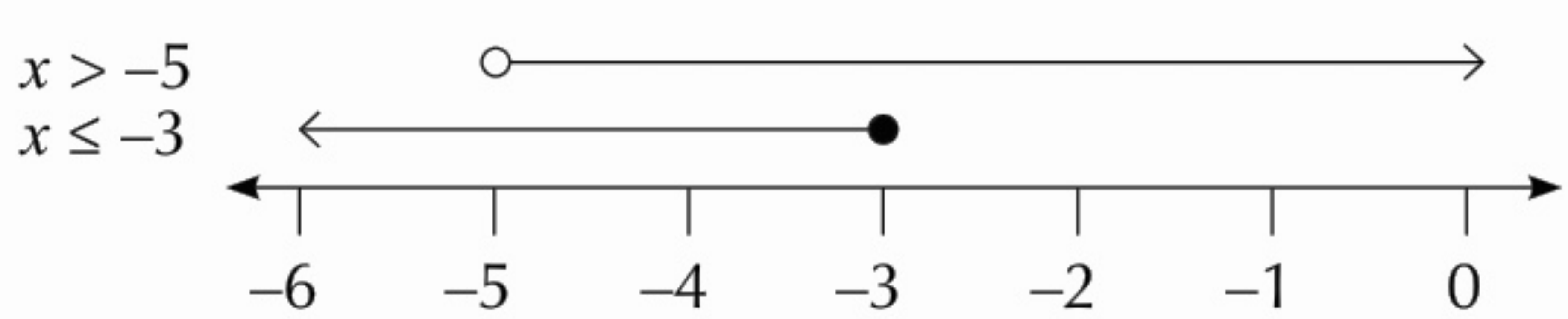
At A-Level, you might need to find values that satisfy **two** inequalities...

**EXAMPLE:** Find the values of $x$ that satisfy both $3x + 4 > 2x - 1$ and $4x \geq 8x + 12$.

Solve each inequality first:

| | |
|---|---|
| $3x + 4 > 2x - 1$ | $4x \geq 8x + 12$ |
| $x + 4 > -1$ | $-4x \geq 12$ |
| $x > -5$ | $x \leq -3$ |

Draw a number line and show both solutions:




Remember, < or > have open circles and ≤ or ≥ have filled-in circles.

The set of values where the lines overlap satisfy both inequalities, so: $\boldsymbol{-5 < x \leq -3}$

## ***Quadratic Inequalities** are a bit **Different***

1) **Quadratic inequalities** have a **squared term** in them — like quadratic equations.
2) Quadratic equations can have two solutions (see p.16). Quadratic inequalities are similar — their solutions can be **two sets of values**, or a **range enclosed by two values**.

**EXAMPLE: a)** Solve $3x^2 - 22 > 5$.

Add 22 to both sides: $3x^2 - 22 + 22 > 5 + 22$
$3x^2 > 27$
Then divide both sides by 3: $3x^2 \div 3 > 27 \div 3$
$x^2 > 9$

So $\boldsymbol{x < -3}$ **or** $\boldsymbol{x > 3}$

You can check your solutions have the right inequality signs by trying some values. E.g. to check $x < -3$, try $x = -4$: $x^2 = 16$ which is $> 9$, so $<$ is correct.

**b)** Solve $4x^2 - 14 \leq 2$.

Add 14 to both sides: $4x^2 - 14 + 14 \leq 2 + 14$
$4x^2 \leq 16$
Divide both sides by 4: $4x^2 \div 4 \leq 16 \div 4$
$x^2 \leq 4$

So $\boldsymbol{-2 \leq x \leq 2}$

# Inequalities

## *Sketch* a *Graph* to Help You Solve *Quadratic Inequalities*

When a **quadratic inequality** also has **$x$-terms** (e.g. $ax^2 + bx + c < d$), it's a bit trickier to solve. The best method is to turn the inequality into an **equation** you can **sketch** — see p.37 for more on sketching quadratics. Then you can solve the inequality by looking for the part of the graph that satisfies it.

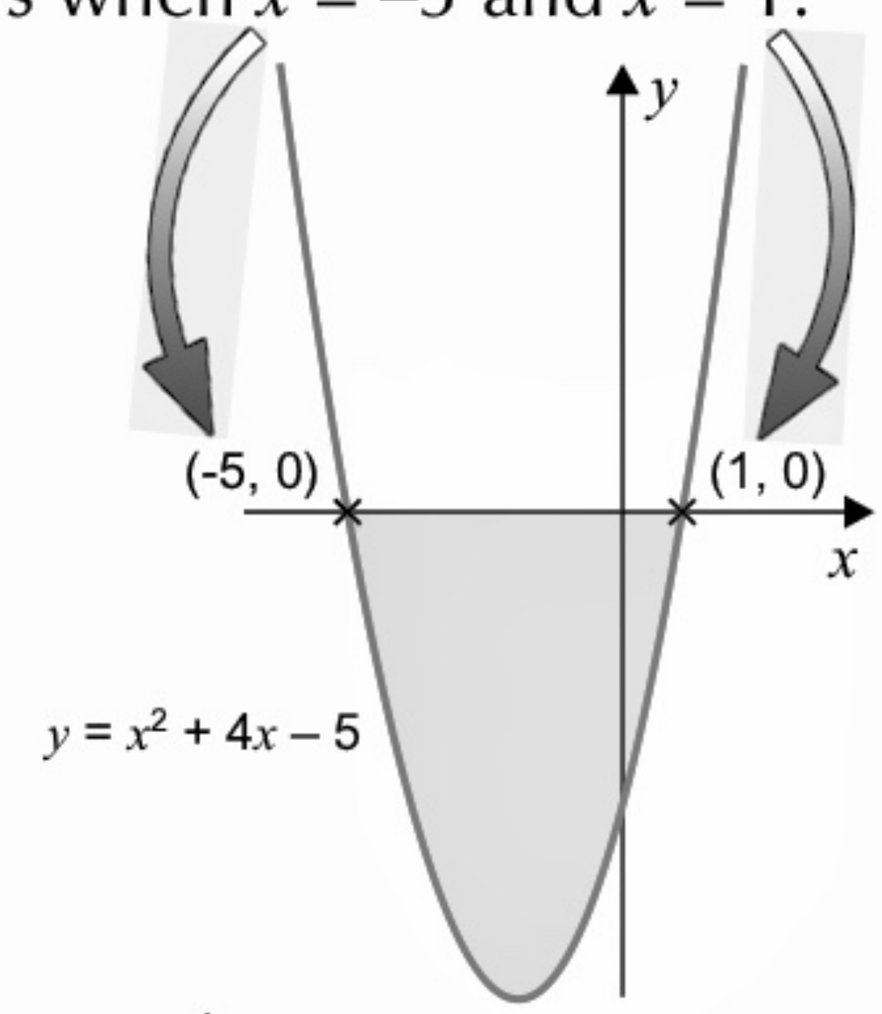
**EXAMPLE: a)** Solve the inequality $-x^2 - 2x + 4 > 2x - 1$.

Rearrange the inequality so that you have 0 on one side: $x^2 + 4x - 5 < 0$

Sketch the graph of $y = x^2 + 4x - 5$.
The coefficient of $x^2$ is positive, which means the graph is u-shaped.

It intersects the $x$-axis when $y = 0$, so solve $x^2 + 4x - 5 = 0$ by factorising:
$x^2 + 4x - 5 = (x + 5)(x - 1) = 0$, so the graph intersects the $x$-axis when $x = -5$ and $x = 1$.

Now you can use the graph to solve the inequality.
You're looking for $x^2 + 4x - 5 < 0$ — that's where the graph is below the $x$-axis. You can see from the $x$-intercepts that this is between the points $(-5, 0)$ and $(1, 0)$.
So the solution is $\mathbf{-5 < x < 1}$.




**b)** Find the values of $x$ which satisfy both $-x^2 - 2x + 4 > 2x - 1$ and $9x \leq 4x - 5$.

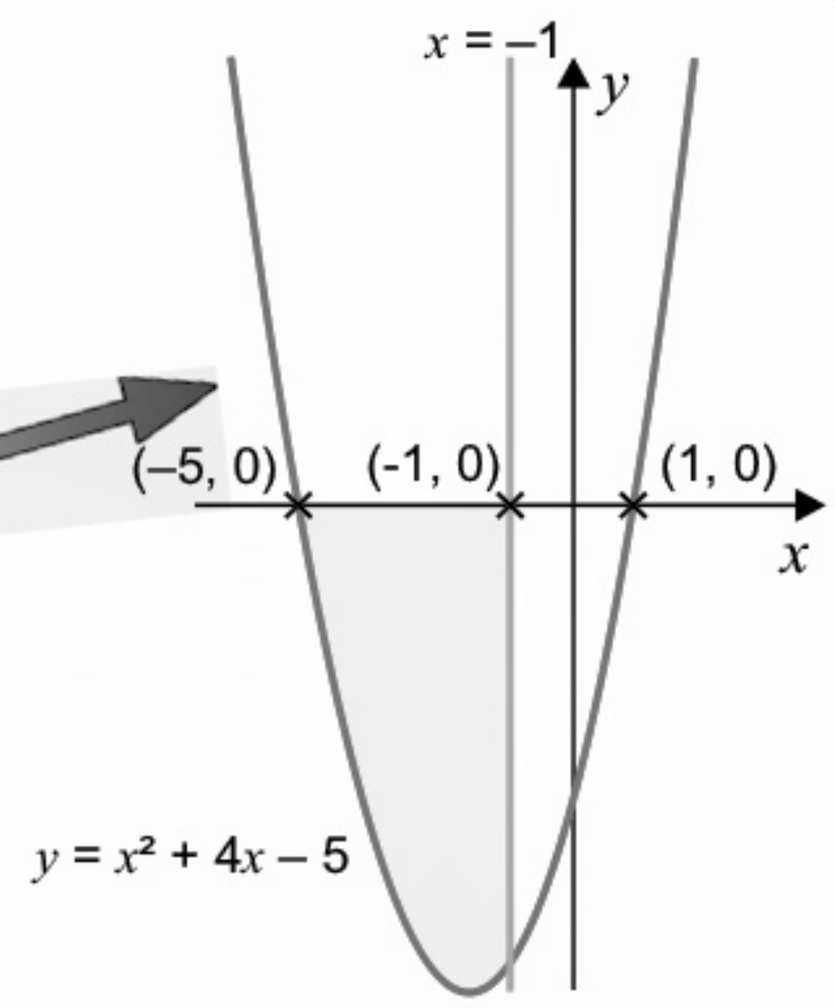
Solve the second inequality in the normal way:
Subtract $4x$ from both sides: $9x - 4x \leq 4x - 5 - 4x$
$5x \leq -5$

Then divide both sides by 5: $5x \div 5 \leq -5 \div 5$
$x \leq -1$

To solve both inequalities, add the line $x = -1$ to the sketch of $y = x^2 + 4x - 5$.

You're looking for where $x^2 + 4x - 5 < 0$ **and** $x \leq -1$ — so where the curve is below the $x$-axis, **and** to the left of $x = -1$.

So the set of values is $\mathbf{-5 < x \leq -1}$.




There's more on showing inequalities using graphs on the next couple of pages.

## *I'm no Picasso — but I can sketch a mean quadratic...*

1) Solve the following linear inequalities:
   a) $4 - 8x \leq -10x - 6$ b) $-13 \leq 2x - 3 < 11$ c) $-11 < 1 - 3x < 7$
2) Use a number line to find the values of $x$ which satisfy both $2x + 4 < 3x$ and $5 \geq 7x - 2$.
3) Solve the following quadratic inequalities:
   a) $3x^2 - 8 < 4$ b) $2x - x^2 + 8 > 0$ c) $6x \geq 2x^2 + 4$ d) $-5x^2 + 4x + 2 < 6x - x^2$
4) Find the values of $x$ which satisfy both $x^2 + 3x + 7 < 3 - 2x$ and $2(3 - 2x) > 14$.

# Graphical Inequalities

## Follow These Four Easy Steps to Show Inequalities on a Graph

1) Write each inequality as an **equation**.

   **Replace** the **inequality sign** with an **= sign** and write in the form "$y = ...$" if you need to.

2) Draw the **graph** of each equation.

   Use a **dotted line** if the inequality had a $<$ or $>$ sign and a **solid line** if it had a $\leq$ or $\geq$ sign.

3) Decide **which side** of each line is the correct one.

   Put the coordinates of a point into each inequality.
   If the inequality is **true**, the point is on the **correct side** of the line.

   Use the origin if possible.

4) **Shade** or **label** the correct region.

   Check carefully what you're asked to do in the question.

**EXAMPLE:** Draw and shade the region that satisfies all three inequalities below.

$3x > 6 + y$ $\qquad -2 + y \leq x$ $\qquad y \geq -x$

1) Write each inequality as an equation in the form '$y =$':

   $y = 3x - 6$ $\qquad y = x + 2$ $\qquad y = -x$

2) Then draw the graphs. They're all linear equations, so they're just straight lines:

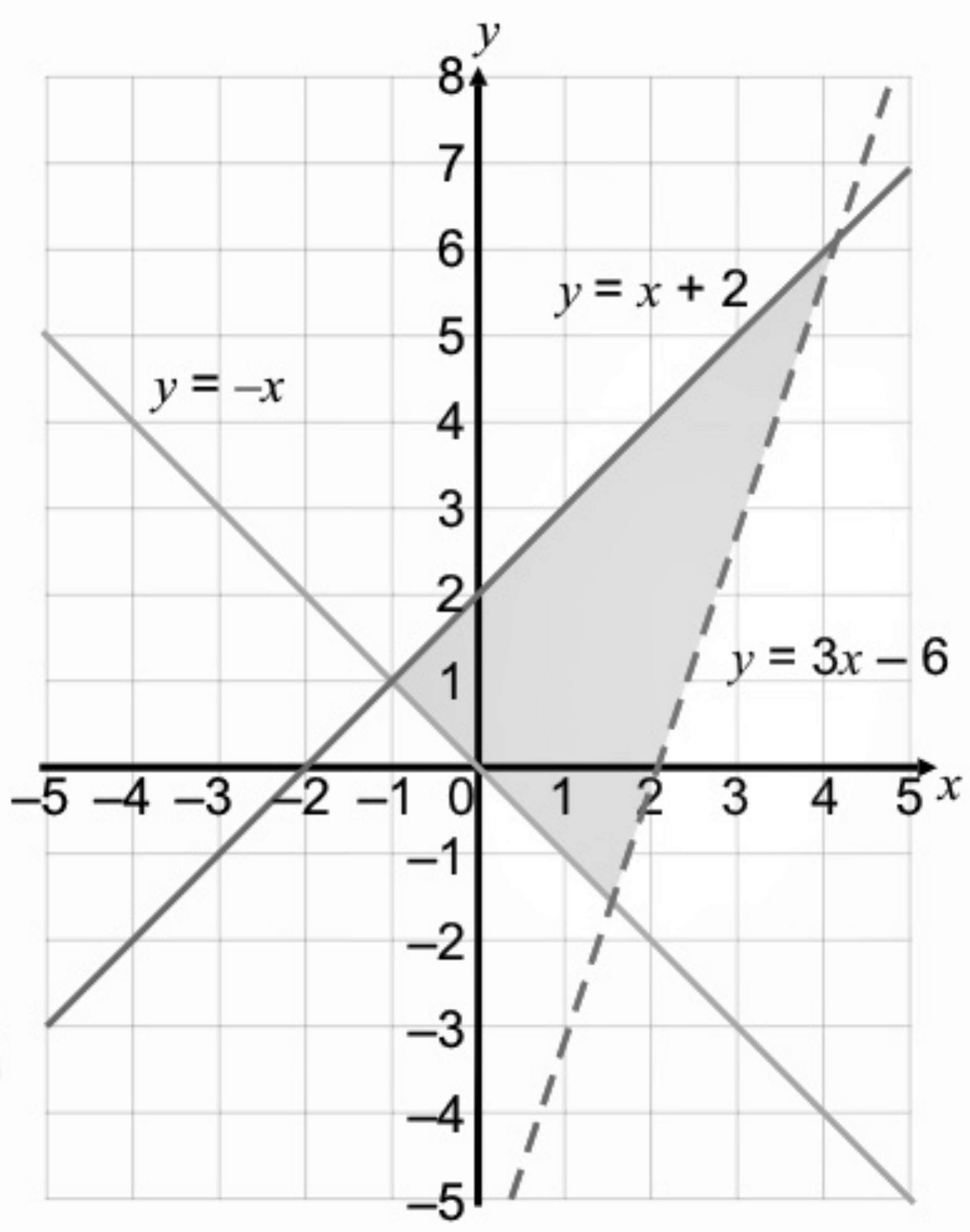
3) Decide which side of each line is correct by plugging some coordinates into each inequality. You can use the origin for $3x > 6 + y$ and $-2 + y \leq x$, but the origin is on the line $y = -x$ so use (1, 2) for $y \geq -x$:

   $3x > 6 + y \Rightarrow 0 > 6$ which is false.
   So the origin is on the wrong side of the line.

   $-2 + y \leq x \Rightarrow -2 \leq 0$ which is true.
   So the origin is on the correct side of the line.

   $y \geq -x \Rightarrow 2 \geq -1$ which is true.
   So (1, 2) is on the correct side of the line.

4) Shade the region below $y = 3x - 6$ and $y = x + 2$, and above $y = -x$.




## Shaded regions — good for showing inequalities and avoiding sunburn...

1) For each inequality below, draw a graph and shade the region that satisfies the inequality.

   a) $5x + y > 17$ $\qquad$ b) $3 - y \geq 4x + 2$ $\qquad$ c) $x - 8 < 2y$

2) Shade the region that satisfies the following inequalities:

   $x < 8$ $\qquad x + y > 2$ $\qquad x - 4 < y + 6$

# Graphical Inequalities

## You can Show *Quadratic Inequalities* on a Graph

At A-Level, you'll need to show **quadratic inequalities** on a graph as well as linear ones. To do this, you'll have to accurately sketch a **quadratic graph** using its $x$- and $y$-intercepts and turning point. Have a look at p.37 for how — the examples below will just tell you the key points.

**EXAMPLE:** Draw and label the region that satisfies the inequalities $y > x^2 - 4$ and $3x + 1 \geq y$.

Write the inequalities as equations in terms of $y$:

$y = x^2 - 4 \qquad y = 3x + 1$

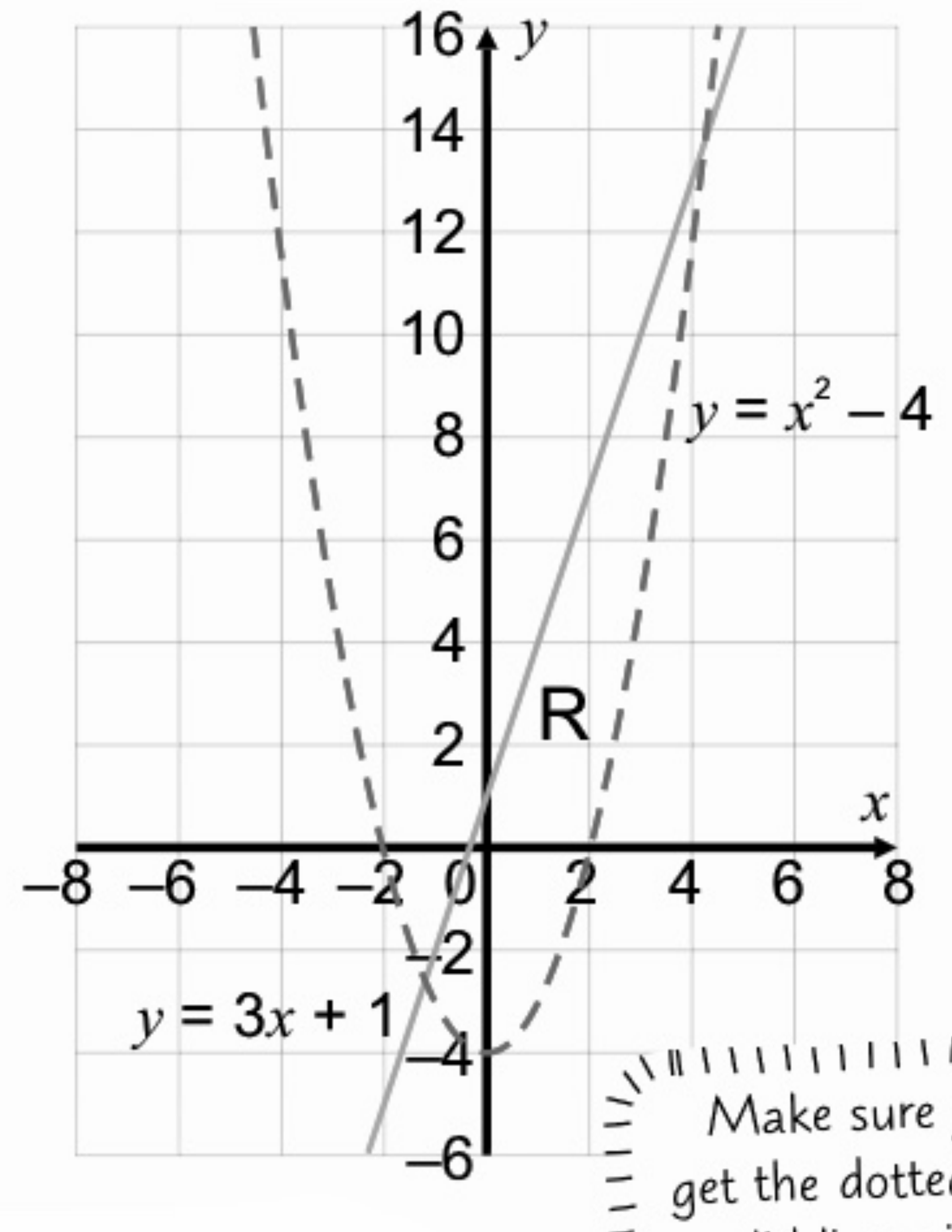
Then draw the graphs. The quadratic graph is u-shaped. It intersects the $y$-axis at $y = -4$, and the $x$-axis at $x = 2$ and $x = -2$. The turning point is at $(0, -4)$.

Plug the origin (0, 0) into each inequality:

$y > x^2 - 4 \Rightarrow 0 > -4$ which is true.
So the origin is on the correct side of the line.

$3x + 1 \geq y \Rightarrow 1 \geq 0$ which is true.
So the origin is on the correct side of the line.

Label the region above $y = x^2 - 4$ and below $y = 3x + 1$.




Make sure you get the dotted and solid lines right.

**EXAMPLE:** Draw and shade the region satisfying the following inequalities:

$-x^2 + 3x - 2 > y \qquad x^2 < y + 9$

Write each inequality as an equation in the form '$y =$':

$y = -x^2 + 3x - 2 \qquad y = x^2 - 9$

Then draw the graphs. $y = -x^2 + 3x - 2$ is n-shaped. It intersects the $y$-axis at $y = -2$ and the $x$-axis at $x = 1$ and $x = 2$. The turning point is at $(1.5, 0.25)$.

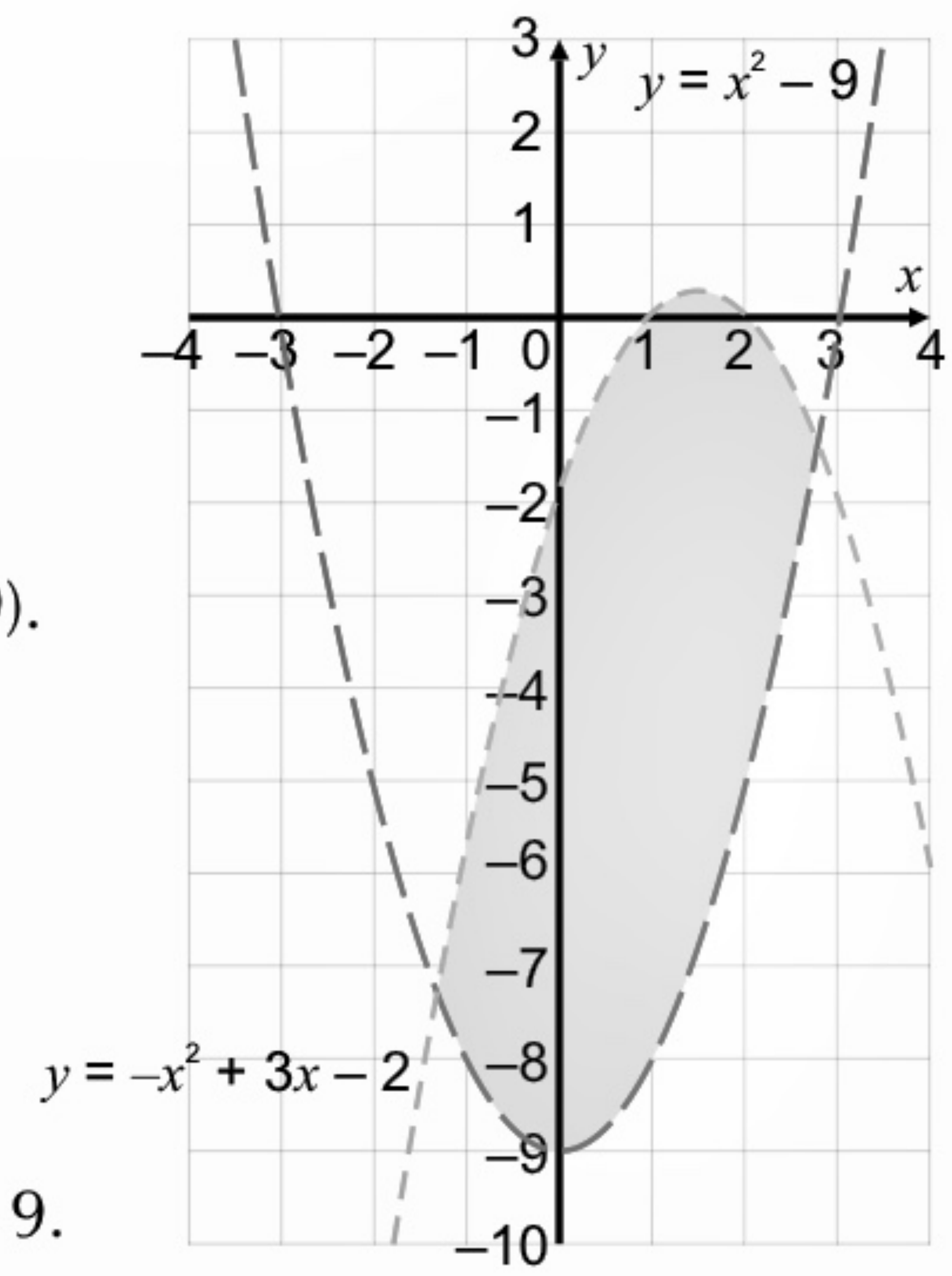
$y = x^2 - 9$ is u-shaped. It intersects the $y$-axis at $y = -9$ and the $x$-axis at $x = 3$ and $x = -3$. The turning point is at $(0, -9)$.

Find the region by substituting (0, 0) into each inequality.

$-x^2 + 3x - 2 > y \Rightarrow -2 > 0$ which is false.
So the origin is on the wrong side of the line.

$x^2 < y + 9 \Rightarrow 0 < 9$ which is true.
So the origin is on the correct side of the line.

Shade the region below $y = -x^2 + 3x - 2$ and above $y = x^2 - 9$.




## *Alternatively, cook the inequalities a roast dinner — that'll satisfy 'em...*

1) Draw and shade the regions that satisfy the following inequalities:
   a) $y > x^2$ and $8x + 4 < 2y$
   b) $-x^2 + 4 \geq y$ and $y + 10 < 14 - x$
   c) $8 - 2x^2 - y < 0$ and $6x > 3y - 18$
   d) $x^2 + 4y > 0$ and $x^2 - 6 < y - 2$

2) Draw and shade the region that satisfies the following inequalities:

$y < 3 + 5x - 2x^2 \qquad y \geq x^2 - x - 2$

# Simultaneous Equations

## *Use **Elimination** when the Simultaneous Equations are **Linear***

1) To solve a pair of **simultaneous equations**, you need to find the values of the **variables** (e.g. $x$ and $y$) which will make **both** equations **true**.
2) For example, the simultaneous equations $x + 2y = 5$ and $3x - y = 1$ have the solution $\boldsymbol{x = 1}$ **and** $\boldsymbol{y = 2}$ because plugging these values into the LHS of the equations gives the RHS.

Here's the **method** for solving simultaneous equations if both equations are **linear**:

1) Rearrange the equations into the form $\boldsymbol{ax + by = c}$.
2) Make the **coefficients** of one of the variables the same by **multiplying** one (or both) of the equations by something.
3) **Add** or **subtract** the equations to get rid of the terms with the same coefficient, then **solve** to find the value of the **remaining variable**.
4) Plug this value back into one of the **original equations** to find the **other variable**.
5) Put the values of **both variables** into the **other original equation** to check your answer.

In step 2, you can ignore the signs of the coefficients. E.g. you could make them 4 and –4.

**EXAMPLE:** Solve the simultaneous equations $3x - 2y = 1$ and $2x + 3y = 11.5$.

Both equations are already in the form $ax + by = c$. Label them (1) and (2).

(1) $3x - 2y = 1$ (2) $2x + 3y = 11.5$

Make the coefficients of one of the variables the same.
Label the new equations (3) and (4):

(1) × 3 → $9x - 6y = 3$ (3)

(2) × 2 → $4x + 6y = 23$ (4)

Now eliminate the $y$'s to find the value of $x$ by adding the two equations together:

(3) + (4) $13x = 26 \Rightarrow x = 2$

If the signs are the same, then subtract.
If the signs are opposites, then add.

Putting this value of $x$ into either equation will allow you to solve for $y$.

$x = 2$ in (2) gives $2x + 3y = 11.5$
$4 + 3y = 11.5$
$3y = 7.5 \Rightarrow y = 2.5$

Now check your answer by putting both these values into equation (1).

$x = 2$ and $y = 2.5$ in (1) gives $(3 \times 2) - (2 \times 2.5) = 1$ ✓

So the solution is $\boldsymbol{x = 2}$, $\boldsymbol{y = 2.5}$.

## *Eliminating the x's — seems a tad cruel...*

1) Solve the following simultaneous equations:

a) $y = 4x - 1$
$5x - 2y = 5$

b) $3x + 8y = 25$
$12x - 10y = 16$

c) $3x - 2y = 8$
$7x - 5y = 17$

d) $\frac{1}{2}x + 2y = 12$
$y = 4x - 11$

# Simultaneous Equations

## Use **Substitution** When One Equation is **Non-Linear**

If there are any squared terms in the equations, you can't use the elimination method — you have to use the **substitution** method instead. You might have seen this method at GCSE, but you'll use it at **A-Level** with trickier equations, e.g. with more than one squared variable.

1) Rearrange the **linear equation** to get a variable **on its own**.
2) **Substitute** this into the quadratic equation.
3) Rearrange into the **standard format** for a quadratic equation ($ax^2 + bx + c = 0$) and solve (see p.16-18).
4) Plug these values back into the **linear equation** to find the values of the **other variable**.
5) Put the values of **both variables** into the **quadratic equation** to check your answer.

At GCSE, you might have rearranged the quadratic equation instead. But you can't always get a variable on its own in the quadratic (see below), so I've said to rearrange the linear one here.

**EXAMPLE:** Solve the simultaneous equations $2y = 5 - x$ and $x^2 + y^2 = 10$.

Rewrite the linear equation so either $x$ or $y$ is by itself on one side of the equation. Label the equations (1) and (2).

(1) $x = 5 - 2y$

(2) $x^2 + y^2 = 10$

Substitute (1) into (2) to form another equation — call this (3).

$x^2 + y^2 = 10$

$(5 - 2y)^2 + y^2 = 10$ (3)

Then rearrange (3) to make a quadratic equation in the form $ax^2 + bx + c = 0$ and solve it.

$25 - 10y - 10y + 4y^2 + y^2 = 10$ (3)

$5y^2 - 20y + 15 = 0$

$y^2 - 4y + 3 = 0$

$(y - 3)(y - 1) = 0$, so $y = 3$ or $y = 1$

You should factorise to solve the quadratic at this step if possible — you can use the quadratic formula if not.

To find the corresponding values of $x$, put each $y$-value back into the linear equation.

Substitute $y = 3$ into (1) to give: $x = 5 - (2 \times 3) = 5 - 6 = -1$

Substitute $y = 1$ into (1) to give: $x = 5 - (2 \times 1) = 5 - 2 = 3$

Check your answer by putting both pairs of values into equation (2).

$x = -1$ and $y = 3$ in (2) gives $(-1)^2 + 3^2 = 10$ ✓

$x = 3$ and $y = 1$ in (2) gives $3^2 + 1^2 = 10$ ✓

So the solutions are $\boldsymbol{x = -1, y = 3}$ **and** $\boldsymbol{x = 3, y = 1}$.

There are two solutions, so the graphs of these equations will cross in two places.

## Why did the chickens simultaneously cross the road*...

1) Solve the following simultaneous equations:

a) $y = 4x + 4$ and $y = x^2 + 3x - 8$

b) $y^2 - x^2 = 0$ and $3x - y = 3$

c) $x^2 + xy - 10 = 0$ and $y + 2x = -7$

* [EDIT — don't worry, we've since had words with the 'Editor' responsible for that bad joke...]

# Proof

## Remember These *Important Facts* to Help You **Prove Things**

You came across proof at GCSE — but at A-Level you'll have to prove some **trickier** things. So you need to make sure you're really comfortable with the **basic skills**.

These facts are a good starting point in a lot of proofs:

1) You can write any **even number** as $2x$.
2) You can write any **odd number** as $2x + 1$.
3) You can write **consecutive numbers** as $x$, $x + 1$, $x + 2$, etc.
4) You can show something is a **multiple** of a number, $x$, by showing it can be written '$x \times$ something'.
5) When you **add**, **subtract** or **multiply integers**, you'll always end up with an **integer**.

$x$ is just an integer (a whole number).

**EXAMPLE:** Prove that the difference between any two even numbers is always even.

Write the two even numbers as $2x$ and $2y$, where $x$ and $y$ are integers.

Now find the difference: $2x - 2y = 2(x - y)$. The difference between two integers is an integer, so $(x - y)$ is an integer.

$2x - 2y$ can be written as $2n$, where $n = x - y$. **This means that the difference between any two even numbers is a multiple of two, and is therefore even.**

**EXAMPLE:** Prove that, for any integer $a$, the expression $(3a - 1)(a + 2) + 2(a + 3)(a + 2)$ is always a multiple of 5.

Multiply out the brackets: $3a^2 + 5a - 2 + 2(a^2 + 5a + 6) = 3a^2 + 5a - 2 + 2a^2 + 10a + 12$

Combine like terms: $5a^2 + 15a + 10$

Take out a factor of 5: $5(a^2 + 3a + 2)$. As $a$ is an integer, $a^2 + 3a + 2$ is also an integer.

**The expression can be written as $5x$, where $x = (a^2 + 3a + 2)$, so it is a multiple of 5.**

## *You can Use Facts to **Build Up** your **Argument** in a Proof*

In some proofs, you'll need to use **facts** and **laws** from maths to show something is true.

**EXAMPLE:** Prove that the result of dividing a rational number by any other rational number is also rational.

You'll learn more about this type of proof at A-Level — it's called 'proof by deduction'.

Call the two rational numbers $x$ and $y$. Rational numbers can be written as fractions with integers on the top and bottom, so write $x = \frac{a}{b}$ and $y = \frac{c}{d}$ where $a$, $b$, $c$ and $d$ are integers, and $b$, $c$ and $d$ are not 0 (because you can't divide by 0).

$x \div y = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$. $ad$ and $bc$ are the products of two integers, so they are also integers. $bc$ is not 0, as neither $b$ nor $c$ is zero.

**The result of dividing two rational numbers can be written as a fraction with integers on the top and bottom and a non-zero denominator, so it is also rational.**

# Proof

## *You can **Disprove** by Finding a **Counter Example***

Finding a **counter example** is one of the easiest ways to prove a statement is **wrong**. If you can just show **one case** of where the statement doesn't work, then you've disproved it. Sorted.

**EXAMPLE:** Find a counter example to disprove the following statement:
"When $x$ is prime, $2x + 1$ is prime."

Try different primes in the expression.

Substituting $x = 3$ into $2x + 1$ gives $(2 \times 3) + 1 = 7$
7 is a prime number, so try again...

Substituting $x = 7$ into $2x + 1$ gives $(2 \times 7) + 1 = 15$
15 is not prime, so the statement is **not true**.

It can sometimes take a while to find a counter-example, but just keep your cool...

**EXAMPLE:** By finding a counter example, disprove the following statement:
"Given that $a$ and $b$ are integers, if $a > b$, then $a^2 + 1 > b^2 + 1$."

At first glance, this might look correct — but take a closer look at the statement. $a$ and $b$ are integers, which means they can be negative numbers too...

So if $a = 1$ and $b = -2$, then $1 > -2$, so $a > b$.
But $a^2 + 1 = 2$ and $b^2 + 1 = 5$, so $a^2 + 1 < b^2 + 1$.

For $a = 1$ and $b = -2$, the second part of the statement isn't satisfied.

So this statement is **not true**.

**EXAMPLE:** Ian says, "If $x$ and $y$ are two different irrational numbers, then $xy$ must also be irrational."
Find an example to show that Ian is wrong.

The square roots of any non-square number are irrational, so try $x = \sqrt{8}$ and $y = \sqrt{2}$:

$xy = \sqrt{8} \times \sqrt{2} = \sqrt{8 \times 2} = \sqrt{16} = 4$

4 is not an irrational number, so Ian is **wrong**.

You saw on p.6 that an irrational number is one that can't be written exactly as a fraction.

## *I looked for the proof in my pudding last night — I just got really sticky...*

1) Prove that $(n + 1)^2 - (n - 1)^2$ is even for any integer $n$.
2) Prove that the product of any two odd numbers is always odd.
3) Prove that the result of subtracting a rational number from any other rational number is also rational.
4) Phyllis believes the following statement is true: "$2^n + 1$ always gives a prime number, when $n$ is an integer." Prove, by finding a counter example, that Phyllis is wrong.
5) Find a counter example to disprove the following statement:
   "For any pair of integers $x$ and $y$, $\frac{x^2}{y^2} < \frac{x}{y}$."

# Functions

## *Functions Take an Input and Give an Output*

1) A **function** is a type of **mapping** — a set of **instructions** on how to get from **one number** to **another**. There are **two** main ways to write a function:

$$f(x) = 8x + 1$$
$$f : x \to 8x + 1$$

You might not have seen this notation at GCSE, so don't panic if you don't recognise it.

2) These both mean that 'f' takes a value for $x$, **multiplies** it by 8, **adds** 1, and **outputs** the result.
3) The set of values you put into a function is called the **domain**. The set of values output from a function is called the **range**. A function maps **each value** in the **domain** to **one** value in the **range**.

There's more about mappings, domains and ranges in the second year of A-Level Maths.

4) You can **evaluate** a function for different values by substituting each value into the function:

**EXAMPLE: a)** Find f(4) for the function $f(x) = (x + 6)^2$.

Just put $x = 4$ into the function and work out the answer: $f(4) = (4 + 6)^2 = 10^2 = \mathbf{100}$

**b)** $f : x \to 8x - 7$. Find f(–3).

Substitute $x = -3$ into the function:

$f(-3) = (8 \times -3) - 7 = \mathbf{-31}$

**c)** $f(x) = 2x^2 - 1$. Find the value(s) of $x$ when $f(x) = 7$.

Set the function equal to 7 and solve:

$$2x^2 - 1 = 7$$
$$2x^2 = 8$$
$$x^2 = 4 \Rightarrow \boldsymbol{x = -2 \text{ or } x = 2}$$

## *You can Put Functions Together to Make Composite Functions*

1) When you have **two different functions**, say $f(x)$ and $g(x)$, you can combine them to make a **single new function**. This is called a **composite function**.
2) The composite function of $f(x)$ and $g(x)$ is written **fg(*x*)** — which means 'do **g first**, then **apply f** to your answer'. $ff(x)$ or $f^2(x)$ means 'do f **twice**'.
3) You **always** do the function **closest to *x*** first.

Composite functions were covered at GCSE, but there's some trickier stuff to learn about them in Year 2 of A-Level — so make sure you know the basics here.

**EXAMPLE:** $f(x) = x^2 + 20$ and $g(x) = 2x + 3$, find:
**a)** fg($x$)

$fg(x) = f(g(x)) = f(2x + 3) = \mathbf{(2x + 3)^2 + 20}$
(or $4x^2 + 12x + 29$)

**b)** gf($x$)

$$gf(x) = g(f(x))$$
$$= g(x^2 + 20) = 2(x^2 + 20) + 3$$
$$= \mathbf{2x^2 + 43}$$

Usually, you'll find that $fg(x) \neq gf(x)$.

**EXAMPLE:** $f(x) = x^2 + 1$, $g(x) = 4 - 2x$ and $h(x) = 3 + x$. Find fgh(3).

Start with the function closest to $x$ and work outwards. So do h(3) first: $3 + 3 = 6$.

Then apply g to this result: $g(6) = 4 - (2 \times 6) = -8$

Then apply f to this result: $f(-8) = (-8)^2 + 1 = 65$, so $fgh(3) = \mathbf{65}$.

# Functions

## *An **Inverse Function Reverses** the Effect of the **Original Function***

1) The **inverse** of a function **reverses** what the function does.
   The inverse of a function f($x$) is written with the notation $\mathbf{f^{-1}(x)}$.
2) You'll see inverse functions in more detail in Year 2 of A-Level.
3) You can **find the inverse** of a function using these steps:

So if f(2) = 5, then $f^{-1}(5) = 2$.

> 1) Write out the function 'f($x$) = ...' but **replace** f($x$) with $x$, and substitute $y$ into the expression instead of $x$. This gives you $\mathbf{x = f(y)}$.
> 2) **Rearrange** to make $\mathbf{y}$ the **subject**.
> 3) Then **replace** $\mathbf{y}$ with $\mathbf{f^{-1}(x)}$.

**EXAMPLE:** If f($x$) = $4x + 12$, find $f^{-1}(x)$.

Replace f($x$) with $x$ and $x$ with $y$: $x = 4y + 12$

Then rearrange to make $y$ the subject: $4y = x - 12 \Rightarrow y = \frac{x-12}{4}$

Finally, replace $y$ with $f^{-1}(x)$ to get: $\mathbf{f^{-1}(x) = \frac{x-12}{4}}$

If you **combine** a function f($x$) with its **inverse** $f^{-1}(x)$, you get $\mathbf{x}$.

$f^{-1}f(x) = ff^{-1}(x) = x$

This is because f($x$) and $f^{-1}(x)$ just **reverse** each other, so the **output** is the **original input**.
You can use this to check whether you've worked out an inverse function correctly.

**EXAMPLE: a)** If f($x$) = $8 - 12x$, find $f^{-1}(2)$.

Find $f^{-1}(x)$ — replace f($x$) with $x$, and $x$ with $y$:
$x = 8 - 12y$

Rearrange for $y$: $y = \frac{8-x}{12}$

Replace $y$ with $f^{-1}(x)$ to get $f^{-1}(x) = \frac{8-x}{12}$

Substitute $x = 2$ into the function: $\frac{8-2}{12} = \mathbf{\frac{1}{2}}$

**b)** Show that $ff^{-1}(x) = x$ for this function.

You know that $f^{-1}(x) = \frac{8-x}{12}$,
so just put this into f($x$) and simplify.

$ff^{-1}(x) = f\left(\frac{8-x}{12}\right) = 8 - 12\left(\frac{8-x}{12}\right)$
$= 8 - (8 - x) = \mathbf{x}$

$ff^{-1}(x) = x$, so the inverse is correct. Hoorah.

## *These pages : words and numbers → expert function knowledge...*

1) For the function $g : x \to \frac{11x+13}{21-7x}$, evaluate g(2).
2) f($x$) = $5x^2 + 12x - 7$. Evaluate f(3).
3) If f($x$) = $6x + 4$ and g($x$) = $\frac{28}{x}$, find:
   a) f(2) b) g(4) c) fg($x$) d) gf($x$)
4) If f($x$) = $4x^2 + 3$ and g($x$) = $6x - 6$, find:
   a) fg(0) b) gf(0) c) $f^2(0)$ d) $g^2(0)$
5) Find the inverse functions of f($x$) and g($x$) below.
   a) f($x$) = $9 - 11x$ b) g($x$) = $\frac{x-4}{2}$